volume 223

lecture notes in pure and applied mathematics

the Navier-Stokes equations: theory and numerical methods

edited by
Rodolfo Salvi

the Navier-Stokes equations:
theory and numerical methods

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Preface

This volume contains the proceedings of the *International Conference on the Navier-Stokes Equations: Theory and Numerical Methods,* held in Villa Monastero in Varenna, Lecco, Italy. The meeting followed two others of the same type held in Villa Monastero in 1992 and 1997.

Interest in mathematical analysis of nonlinear phenomena has led to increased interdependence and cooperation among scientists. Also, the rapid development of large-scale computers and progress in mathematical analysis has encouraged industrial use of nonlinear mathematical models for solving problems of technology. The attempt to understand nonlinear problems has stimulated application of mathematics in many areas of science, in particular, in fluid mechanics.

There is a real need to promote and realize unifying and distinctive meetings that point out the most important contributions, indicate the directions of future research, and promote collaboration between young and established researchers. The conferences in Varenna moved in the direction of these objectives. By any measure, this event was a great success.

More than 80 invited specialists from all around the world participated. This volume collects 23 selected articles which cover a wide spectrum of topics in the mathematical theory of fluid mechanics: compressible and incompressible fluids, nonnewtonian fluids, the free-boundary problem, qualitative properties of solutions, hydrodynamic potential theory, and numerical experiments. The contributions present original results and surveys on recent developments.

In addition to the editor, the organizers of the conference were M. Biroli (Politecnico di Milano), J. G. Heywood (University of British Columbia), K. Masuada (Meiji University), G. Nespoli (Politecnico di Milano-Sede di Lecco), R. Rautmann (University of Paderborn), and V. A. Solonnikov (University of St. Petersburg).

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Rodolfo Salvi

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A Reynolds equation derived from the Micropolar Navier-Stokes system

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Abstract

In this paper we generalize [3] to the nonlinear case.

1 Introduction

We consider the following system which is a mathematical model of stastionary flow of viscous micropolar fluids, which describes the motion of solid particle suspension in a liquid:

$$-(\nu + \nu_\gamma)\,\Delta u + (u \cdot \nabla)\,u + \nabla p = 2\nu_\gamma \; rot\ \omega \quad \text{in } Q \tag{1.1}$$

$$div\, u = 0 \qquad \text{in } Q \tag{1.2}$$

$$-(c_a + c_d)\,\Delta\omega + I\,(u \cdot \nabla)\,\omega - (c_o + c_d - c_a)\,\nabla\, div\, \omega +$$

$$+ 4\nu_\gamma\omega == 2\nu_\gamma\, rot\, u \;\text{ in } Q \tag{1.3}$$

$$u = g \quad \text{and} \quad \omega = f \qquad \text{on } \partial Q. \tag{1.4}$$

where ν is the usual kinematic Newtonian viscosity, ν_γ is the dynamic microrotation viscosity, c_0, c_a, c_d are new viscosities connected with asymmetricity of the stress tensor, called coefficients of angular viscosities, u is the velocity vector, p the pressure, ω is the angular velocity of rotation of the particules of the fluid, I is a scalar called the microinertia coefficient.

It has been extensively examined from various points of view, both mechanical and mathematical, see for exemple [5], [13], [12], [4], [8], [9].

We consider the problem (1.1)-(1.4) viewed as a lubrication thin film problem describing flows of micropolar fluids in an infinitely long journal bearing :

$$Q = \{(x_1, x_2, x_3\ ) \in \boldsymbol{R}^3 (x_1, x_2) \in \Omega^\varepsilon,\ -\infty < x_3 < +\infty\}$$

where

$$\Omega^\varepsilon = \left\{(x_1,x_2) \in \boldsymbol{R}^2 : 0 < x_1 < 2\pi\,,\ 0 < x_2 < \varepsilon\, h(x)\right\}, \quad \varepsilon > 0\,,$$

and h is a positive smooth function on the interval $[0, 2\pi]$. Then Ω^ε may be viewed as a cross section of Q perpendicular to the $0x_3$ axis.

Assuming that the flow in Q dœs not depend on the x_3 coordinate (it is the same in each cross section $x_3 = const$), the velocity component u_3 in the direction $0x_3$ is zero, and the axes of rotation of particles are parallel to $0x_3$. Then the fields u, ω, p reduce to

$$u = (u_1\,(x_1, x_2),\ u_2\,(x_1, x_2), 0)\ ,\ \omega = (0,\ 0,\ \omega_3(x_1, x_2\ ))\ ,\ p(x) = p(x_1, x_2)$$

Putting u, ω, p in the above form in system (1.1)-(1.3) we obtain the following two dimensional nonlinear problem in Ω^ε:

$$-\left(\nu + \nu_\gamma\right) \Delta u_1^\varepsilon + \frac{\partial p^\varepsilon}{\partial x_1} = 2\nu_\gamma \frac{\partial \omega_3^\varepsilon}{\partial x_2} - u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial x_1} - u_2^\varepsilon \frac{\partial u_1^\varepsilon}{\partial x_2} \tag{1.5}$$

$$-\left(\nu + \nu_\gamma\right) \Delta u_2^\varepsilon + \frac{\partial p^\varepsilon}{\partial x_2} = -2\nu_\gamma \frac{\partial \omega_3^\varepsilon}{\partial x_1} - u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x_1} - u_2^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x_2} \tag{1.6}$$

$$\frac{\partial u_1^\varepsilon}{\partial x_1} + \frac{\partial u_2^\varepsilon}{\partial x_2} = 0 \tag{1.7}$$

$$-\left(c_a + c_d\right) \Delta \omega_3^\varepsilon + 4\nu_\gamma \omega_3^\varepsilon = 2\nu_\gamma \left(\frac{\partial u_2^\varepsilon}{\partial x_1} - \frac{\partial u_1^\varepsilon}{\partial x_2}\right) - I\left(u_1^\varepsilon \frac{\partial \omega_3^\varepsilon}{\partial x_1} + u_2^\varepsilon \frac{\partial \omega_3^\varepsilon}{\partial x_2}\right). \tag{1.8}$$

The appropriate Dirichlet nonhomogeneous boundary conditions of (1.4) are :

$$u^\varepsilon = g^\varepsilon \qquad \text{on } \partial\Omega^\varepsilon, \qquad \int_{\partial\Omega^\varepsilon} g^\varepsilon.n\ d\sigma = 0, \tag{1.9}$$

$$\omega_3^\varepsilon = f^\varepsilon \quad \text{ on } \partial\Omega^\varepsilon. \tag{1.10}$$

In Section 2 we introduce the notations and define the weak solution of the microscopic problems (1.5-1.10) in narrow films Ω^ε, $\varepsilon > 0$. To study behaviour of solutions $(u_1^\varepsilon,\ u_2^\varepsilon,\ p^\varepsilon,\ \omega_3^\varepsilon)$ of problems (1.5-1.10) as $\varepsilon \to 0$ we rescale them by introducing the new independent variable $y = x_2/\varepsilon$. We obtain in this way a family of problems in the fixed domain

$$\Omega = \left\{(x_1, y) \in \boldsymbol{R}^2 : 0 < x_1 < 2\pi\,,\ 0 < y < h(x_1)\right\},$$

for some new dependent variables $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{p}^\varepsilon, \widehat{\omega}_3^\varepsilon)$. Of course, it is crucial to know the relations between the boundary data g^ε and f^ε for various ε. At the end of this Section we specify these functions as well as the variables $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{p}^\varepsilon, \widehat{\omega}_3^\varepsilon)$ in Ω in terms of solutions $(u_1^\varepsilon, u_2^\varepsilon, p^\varepsilon, \omega_3^\varepsilon)$ in Ω^ε.

In Section 3, we obtain uniform estimates of solutions $\left(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{p}^\varepsilon, \widehat{\omega}_3^\varepsilon\right)$ with respect to ε. To this end we have to specify the behaviour of the coefficients in (1.5)-(1.10) as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ in variational formulations of the rescaled problems, we obtain the following macroscopic laws for the related limit functions $(u_1^*, u_2^*, p^*, \omega_3^*)$:

$$\frac{\partial p^*}{\partial x_1} = \frac{\partial^2 u_1^*}{\partial y^2} + 2N_o^2 \frac{\partial \omega_3^*}{\partial y} \qquad \text{in} \quad \mathcal{D}'(\Omega) \tag{1.11}$$

$$\frac{\partial p^*}{\partial y} = 0 \qquad \text{in} \quad \mathcal{D}'(\Omega) \tag{1.12}$$

$$\frac{\partial^2 \omega_3^*}{\partial y^2} = 2N_o^2 \left(\omega_3^* + \frac{\partial u_1^*}{\partial y}\right) \qquad \text{in} \quad \mathcal{D}'(\Omega). \tag{1.13}$$

Then, as in [3] this system (1.11)-(1.13) can be solved explicitly in $u_1^*(x_1, y)$, $\omega_3^*(x_1, y)$, using the continuity equation in the form

$$\int_0^{h(x_1)} u_1^*(x_1, y)\, dy = g(x_1), \quad 0 < x_1 < 2\pi$$

where g is a known function, to obtain the following Reynolds equation for p^* :

$$\nabla \left\{ \frac{h^3(x)}{1 - N_o^2} \Phi\left(h(x), N_o\right) \nabla p^*(x) \right\} = \frac{1}{2} s \frac{\partial h(x)}{\partial x_1}, \tag{1.14}$$

where $0 \leq N_o < 1$ and

$$\Phi(h, N_o) = \frac{1}{12\nu} + \frac{1}{4h^2(1 - N_o^2)} - \frac{1}{4h}\sqrt{\frac{N_o^2}{1 - N_o^2}} cth\left(hN_o\sqrt{1 - N_o^2}\right). \tag{1.15}$$

2 Weak formulation of the problem (1.5-1.10).

We denote by $V(\Omega^\varepsilon)$ the subspace of $(H_0^1(\Omega^\varepsilon))^2$ defined by

$$V(\Omega^\varepsilon) = \{ u \in (H_0^1(\Omega^\varepsilon))^2 \,,\ div\, u = 0 \}, \quad L_0^2(\Omega^\varepsilon) = \{p \in L^2(\Omega^\varepsilon) \int_{\Omega^\varepsilon} p = 0\}.$$

It is known [6] that for any functions : $g^\varepsilon = (g_1^\varepsilon, g_2^\varepsilon) \in (H^{1/2}(\partial\Omega^\varepsilon))^2$ with

$$\int_{\partial\Omega^\varepsilon} g^\varepsilon . n \, d\sigma = 0 \qquad \text{and} \qquad f^\varepsilon \in H^{1/2}(\partial\Omega^\varepsilon)$$

there exist functions $G^\varepsilon = (G_1^\varepsilon, G_2^\varepsilon) \in (H^1(\Omega^\varepsilon))^2$ with $div\, G^\varepsilon = 0$ in Ω^ε, and $F^\varepsilon \in H^1(\Omega^\varepsilon)$ such that $G^\varepsilon = g^\varepsilon$ and $F^\varepsilon = f^\varepsilon$ on $\partial\Omega^\varepsilon$.

Problem (P^ε): (*The weak formulation of problem (1.5-1.10)*) Let

$$G^\varepsilon = (G_1^\varepsilon, G_2^\varepsilon) \in (H^1(\Omega^\varepsilon))^2 \cap (L^\infty(\Omega^\varepsilon))^2 \text{ , with } div\, G^\varepsilon = 0 \tag{2.1}$$

$$F^\varepsilon \in H^1(\Omega^\varepsilon) \cap L^\infty(\Omega^\varepsilon). \tag{2.2}$$

Find u_1, u_2, ω_3 in $H^1(\Omega^\varepsilon)$, and p in $L_0^2(\Omega^\varepsilon)$ such that for all functions φ, ψ, η in $H_0^1(\Omega^\varepsilon)$,

$$\begin{aligned}(\nu + \nu_\gamma)\int_{\Omega^\varepsilon}\left(\frac{\partial u_1}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\frac{\partial \varphi}{\partial x_2}\right) - \int_{\Omega^\varepsilon} p\frac{\partial \varphi}{\partial x_1} = \\ = 2\nu_\gamma \int_{\Omega^\varepsilon}\frac{\partial \omega_3}{\partial x_2}\varphi - \int_{\Omega^\varepsilon}(u_1\frac{\partial u_1}{\partial x_1} + u_2\frac{\partial u_1}{\partial x_2})\varphi\end{aligned} \tag{2.3}$$

$$\begin{aligned}(\nu + \nu_\gamma)\int_{\Omega^\varepsilon}\left(\frac{\partial u_2}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial \psi}{\partial x_2}\right) - \int_{\Omega^\varepsilon} p\frac{\partial \psi}{\partial x_2} \\ = -2\nu_\gamma \int_{\Omega^\varepsilon}\frac{\partial \omega_3}{\partial x_1}\psi - \int_{\Omega^\varepsilon}(u_1\frac{\partial u_2}{\partial x_1} + u_2\frac{\partial u_2}{\partial x_2})\psi\end{aligned} \tag{2.4}$$

$$\begin{aligned}(c_a + c_d)\int_{\Omega^\varepsilon}\left(\frac{\partial \omega_3}{\partial x_1}\frac{\partial \eta}{\partial x_1} + \frac{\partial \omega_3}{\partial x_2}\frac{\partial \eta}{\partial x_2}\right) + 4\nu_\gamma\int_{\Omega^\varepsilon}\omega_\varepsilon \cdot \eta = \\ = 2\nu_\gamma \int_{\Omega^\varepsilon}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\eta - I\int_{\Omega^\varepsilon}(u_1\frac{\partial \omega_3}{\partial x_1} + u_2\frac{\partial \omega_3}{\partial x_2})\eta\end{aligned} \tag{2.5}$$

and

$$(u_1 - G_1^\varepsilon,\, u_2 - G_2^\varepsilon) \in V(\Omega^\varepsilon) \tag{2.6}$$

$$\omega_3 - F^\varepsilon \in H_0^1(\Omega^\varepsilon), \tag{2.7}$$

Let $J = (J_1, J_2)$ such that $DivJ = 0$ in $(H^1(\Omega))^2$ and $K \in H^1(\Omega)$. We define, for (x_1, x_2) in Ω^ε:

$$G_1^\varepsilon(x_1, x_2) = J_1\left(x_1, \frac{x_2}{\varepsilon}\right), \quad G_2^\varepsilon(x_1, x_2) = J_2\left(x_1, \frac{x_2}{\varepsilon}\right) \tag{2.8}$$

$$F^\varepsilon(x_1, x_2) = \frac{1}{\varepsilon}K\left(x_1, \frac{x_2}{\varepsilon}\right) \tag{2.9}$$

Obviously, G^ε is such that $DivG^\varepsilon = 0$ in $(H^1(\Omega^\varepsilon))^2$ and $F^\varepsilon \in H^1(\Omega^\varepsilon)$ for each $\varepsilon > 0$. Formulas (2.8), (2.9) establish relations between boundary data for different $\varepsilon > 0$.

Introducing from now on the new independent variable $y = \frac{x_2}{\varepsilon}$, we obtain a family of problems in the fixed domain :

$$\Omega = \left\{(x_1, y) \in \mathbb{R}^2 : 0 < x_1 < 2\pi\,,\, 0 < y < h(x_1)\right\},$$

for some new dependent variables

$$\Gamma_0 = \partial\Omega \cap \{y = 0\}, \quad \Gamma_1 = \partial\Omega \cap \{y = h(x_1)\},$$
$$\Gamma_L = (\partial\Omega \cap \{x_1 = 0\}) \cup (\partial\Omega \cap \{x_1 = 2\pi\}).$$

Let u_1, u_2, ω_3, p be a solution of problem (P^ε) and let $(x_1, x_2) = (x_1, \varepsilon y)$ in Ω^ε. We define the following functions in Ω :

$$\widehat{u}_1\,(x_1, y) = u_1\,(x_1, x_2) \quad , \quad \widehat{u}_2\,(x_1, y) = \frac{1}{\varepsilon} u_2\,(x_1, x_2)$$

$$\widehat{\omega}_3\,(x_1, y) = \varepsilon\,\omega_3\,(x_1, x_2) \quad , \quad \widehat{p}\,(x_1, y) = \varepsilon^2\,R_L\;p\,(x_1, x_2)$$

$$\widehat{G}_1\,(x_1, y) = G_1^\varepsilon\,(x_1, \varepsilon\, y)\ ,\ \widehat{G}_2\,(x_1, y) = \frac{1}{\varepsilon} G_2^\varepsilon\,(x_1, \varepsilon\, y)$$

$$\widehat{F}\,(x_1, y) = \varepsilon\, F^\varepsilon\,(x_1, \varepsilon\, y) \quad \text{for } (x_1, y) \in \Omega\,,$$

where $R_L = (\nu + \nu_\gamma)^{-1}$. We set also $N = \left(\frac{\nu_\gamma}{\nu + \nu_\gamma}\right)^{1/2}$ and $R_c = (c_a + c_d)^{-1}$.

Remark 2.1 : Since $div\ G^\varepsilon = 0$ we have $div\ \widehat{G} = 0$. If (u_1, u_2, ω_3, p) is a solution of problem (P^ε) then $(\widehat{u}_1, \widehat{u}_2, \widehat{\omega}_3, \widehat{p})$ is a solution of

Problem $(\widehat{P}^\varepsilon)$: Find $\widehat{u} = (\widehat{u}_1, \widehat{u}_2)$ in $(H^1(\Omega))^2$, $\widehat{\omega}_3$ in $H^1(\Omega)$ and $\widehat{p}$ in $L_0^2\,(\Omega)$ such that

$$(\widehat{u}_1 - \widehat{G}_1,\, \widehat{u}_2 - \widehat{G}_2) \in V(\Omega), \tag{2.10}$$

$$\widehat{u}_1 = \widehat{u}_2 = 0 \text{ on } \Gamma_1\,, \tag{2.11}$$
$$\widehat{u}_1 = s \in \mathbb{R} \text{ on } \Gamma_0\,, \tag{2.12}$$
$$\widehat{u}_2 = 0 \text{ on } \Gamma_0\,,\ \widehat{u}(0, .) = \widehat{u}(2\pi, .) \tag{2.13}$$

$$\widehat{\omega}_3 - \widehat{F} \in H_0^1(\Omega)\ ,\ \widehat{\omega}_3(0, .) = \widehat{\omega}_3(2\pi, .), \tag{2.14}$$

$$\varepsilon^2 \int_\Omega \frac{\partial \widehat{u}_1}{\partial x_1} \frac{\partial \varphi}{\partial x_1} + \int_\Omega \frac{\partial \widehat{u}_1}{\partial y} \frac{\partial \varphi}{\partial y} - \int_\Omega \widehat{p}\, \frac{\partial \varphi}{\partial x_1} =$$
$$= 2N^2 \int_\Omega \frac{\partial \widehat{\omega}_3}{\partial y}\,\varphi \;-\; \varepsilon^2 R_L \int_\Omega (\widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y})\varphi \tag{2.15}$$

$$\varepsilon^4 \int_\Omega \frac{\partial \widehat{u}_2}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \varepsilon^2 \int_\Omega \frac{\partial \widehat{u}_2}{\partial y}\frac{\partial \psi}{\partial y} - \int_\Omega \widehat{p}\frac{\partial \psi}{\partial y} =$$

$$-2N^2\varepsilon^2 \int_\Omega \frac{\partial \widehat{\omega}_3}{\partial x_1}\psi - \varepsilon^4 R_L \int_\Omega (\widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y})\,\psi \tag{2.16}$$

$$\frac{R_L}{R_c}\int_\Omega \frac{\partial \widehat{\omega}_3}{\partial x_1}\frac{\partial \eta}{\partial x_1} + \frac{R_L}{R_c\,\varepsilon^2}\int_\Omega \frac{\partial \widehat{\omega}_3}{\partial y}\frac{\partial \eta}{\partial y} + 4N^2\int_\Omega \widehat{\omega}_3\,\eta =$$

$$= 2N^2\varepsilon^2 \int_\Omega \frac{\partial \widehat{u}_2}{\partial x_1}\eta - 2N^2 \int_\Omega \frac{\partial \widehat{u}_1}{\partial y}\eta - I.R_L \int_\Omega (\widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y})\,\eta \tag{2.17}$$

for all functions $\varphi, \psi, \eta \in H_0^1(\Omega)$.

Following [8] we can establish the following existence, uniqueness, and regularity results :

Theorem 2.1 (Existence): Under the assumptions (2.1),(2.2) and if $(\nu + (1 - C_2)\nu_\gamma) > 0$ then, for every $\varepsilon > 0$ there exists at least one solution to problem $\left(\widehat{P}^\varepsilon\right)$. Where

$$\begin{aligned} C_2 &= (2(1+d^2))^{1/2}(\nu_\gamma^{-1}\|G^\varepsilon\|_{L^4} + 2d(c_a+c_d)^{-1}\|F^\varepsilon\|_{L^4}) + \\ &+ 4\nu_\gamma d^2(c_a+c_d)^{-1}\,. \end{aligned}$$

and d is the diameter of Ω.

Theorem 2.2 (Uniqueness) : Let C_1 and C_4 be the positive constants defined by

$$\begin{aligned} C_1 &= (\nu+\nu_\gamma)\|\nabla G^\varepsilon\| + 2\nu_\gamma\|F^\varepsilon\| + \|G^\varepsilon\|_{L^4}^2 + \\ &+ 2(c_a+c_d)^{-1}\nu_\gamma d\|F^\varepsilon\|_{L^4}\|G^\varepsilon\|_{L^4} + 2\nu_\gamma d\|\nabla F^\varepsilon\| + \\ &+ 8(c_a+c_d)^{-1}d^2{\nu_\gamma}^2\|F^\varepsilon\| + 4(c_a+c_d)^{-1}d\nu_\gamma^2\|G^\varepsilon\|. \end{aligned} \tag{2.18}$$

$$\begin{aligned} C_4 &= (2(1+d^2)\nu^{-1}C_1 + (2(1+d^2))^{1/2}\|G^\varepsilon\|_{L^4} + 4(c_a+c_d)^{-1}d^2\nu_\gamma^2 + \\ &+ 4(c_a+c_d)^{-1}d\nu_\gamma(1+d^2)C_3 + 2d\nu_\gamma(c_a+c_d)^{-1}(2(1+d^2))^{1/2}\|F^\varepsilon\|_{L^4}. \end{aligned}$$

If we assume that $\nu + \nu_\gamma > C_4$, then the solution of problem $\left(\widehat{P}^\varepsilon\right)$ is unique.

Theorem 2.3 (Regularity) Under the assumptions of Theorem 2.1, Theorem 2.2, and if Ω is an open Lipschitz-continuous subset of $\mathbb{R}^2$ ($\partial\Omega \in C^{0,1}$), we have $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ in $(H^2(\Omega))^2$, p in $H^1(\Omega)$ and ω_3^ε in $H^2(\Omega)$.

3 Some estimates and convergence results.

Lemma 3.1 : Assume h satisfying $h(0) = h(2\pi)$. Under the assumptions (2.10)-(2.14) we have

$$\int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y}\right) \widehat{\omega}_3 = 0. \tag{3.1}$$

Proof: We have

$$\int_\Omega \widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1} \widehat{\omega}_3 = \int_{\partial\Omega} \widehat{u}_1 \widehat{\omega}_3^2 \cos(\vec{n}, x_1) d\sigma - \int_\Omega \widehat{\omega}_3^2 \frac{\partial \widehat{u}_1}{\partial x_1} - \int_\Omega \widehat{\omega}_3 \widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1}, \tag{3.2}$$

$$\int_\Omega \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y} \widehat{\omega}_3 = \int_{\partial\Omega} \widehat{\omega}_3 (\widehat{u}_2 \widehat{\omega}_3) \cos(\vec{n}, y) d\sigma - \int_\Omega (\widehat{\omega}_3)^2 \frac{\partial \widehat{u}_2}{\partial y} - \int_\Omega \widehat{\omega}_3 \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y} \tag{3.3}$$

adding (3.2),(3.3) and using (2.10) we get

$$2 \int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y}\right) \widehat{\omega}_3 =$$

$$\int_{\partial\Omega} (\widehat{\omega}_3)^2 \widehat{u}_1 \cos(\vec{n}, x_1) d\sigma + \int_{\partial\Omega} (\widehat{\omega}_3)^2 \widehat{u}_2 \cos(\vec{n}, y) d\sigma$$

Now the surface integrals give

$$\int_{\partial\Omega} (\widehat{\omega}_3)^2 \widehat{u}_1 \cos(\vec{n}, x_1) d\sigma =$$

$$\int_{\Gamma_1 \cup \Gamma_0} (\widehat{\omega}_3)^2 \widehat{u}_1 \cos(\vec{n}, x_1) d\sigma + \int_{\Gamma_L} (\widehat{\omega}_3)^2 \widehat{u}_1 \cos(\vec{n}, x_1) d\sigma$$

$$\int_{\partial\Omega} (\widehat{\omega}_3)^2 \widehat{u}_2 \cos(\vec{n}, y) d\sigma = \int_{\Gamma_1 \cup \Gamma_0} (\widehat{\omega}_3)^2 \widehat{u}_2 \cos(\vec{n}, y) d\sigma + \int_{\Gamma_L} (\widehat{\omega}_3)^2 \widehat{u}_2 \cos(\vec{n}, y) d\sigma$$

as $\cos(\vec{n}, y) = 0$ on Γ_L and $\widehat{u}_2 = 0$ on $\Gamma_0 \cup \Gamma_1$. We obtain

$$2 \int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{\omega}_3}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{\omega}_3}{\partial y}\right) \widehat{\omega}_3 = \int_{\Gamma_L} (\widehat{\omega}_3)^2 \widehat{u}_1 \cos(\vec{n}, x_1) d\sigma =$$

$$= \int_0^{h(0)} (\widehat{\omega}_3(0, y))^2 \widehat{u}_1(0, y)\, cos(\vec{n}, x_1) dy +$$

$$\int_0^{h(2\pi)} (\widehat{\omega}_3(2\pi, y))^2 \widehat{u}_1(2\pi, y)\, cos(\vec{n}, x_1) dy$$

using the fact that the functions $h, \widehat{u}_1, \widehat{\omega}_3$ are 2π x_1-periodic, (3.1) follows.

Lemma 3.2 : Assume h satisfying $h(0) = h(2\pi)$. Under the assumptions (2.10)-(2.14) we have

$$\int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y} \right) \widehat{u}_1 = 0 \tag{3.4}$$

and

$$\int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y} \right) \widehat{u}_2 = 0. \tag{3.5}$$

Proof. From

$$\int_\Omega \widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} \widehat{u}_1 = \int_{\partial\Omega} \widehat{u}_1^3 \cos(\vec{n}, x_1) - \int_\Omega \widehat{u}_1^2 \frac{\partial \widehat{u}_1}{\partial x_1} - \int_\Omega \widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} \widehat{u}_1 , \tag{3.6}$$

$$\int_\Omega \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y} \widehat{u}_1 = \int_{\partial\Omega} \widehat{u}_2 \widehat{u}_1^2 \cos(\vec{n}, y) - \int_\Omega \widehat{u}_1^2 \frac{\partial \widehat{u}_2}{\partial y} - \int_\Omega \widehat{u}_1 \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y} , \tag{3.7}$$

$$\int_\Omega \widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial x_1} \widehat{u}_2 = \int_{\partial\Omega} \widehat{u}_1 \widehat{u}_2^2 \cos(\vec{n}, x_1) - \int_\Omega \widehat{u}_2^2 \frac{\partial \widehat{u}_1}{\partial x_1} - \int_\Omega \widehat{u}_1 \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial x_1} , \tag{3.8}$$

$$\int_\Omega \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y} \widehat{u}_2 = \int_{\partial\Omega} \widehat{u}_2^3 \cos(\vec{n}, y) - \int_\Omega \widehat{u}_2^2 \frac{\partial \widehat{u}_2}{\partial y} - \int_\Omega \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y} \widehat{u}_2. \tag{3.9}$$

Adding (3.6) and (3.7), and using (2.10) we get

$$2 \int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y} \right) \widehat{u}_1 = \int_{\partial\Omega} \widehat{u}_1^3 \cos(\vec{n}, x_1) + \int_{\partial\Omega} \widehat{u}_1^2 \widehat{u}_2 \cos(\vec{n}, y) =$$

$$= \int_{\Gamma_L} \widehat{u}_1^3 \cos(\vec{n}, x_1) = - \int_0^{h(0)} \widehat{u}_1^3(0, y) + \int_0^{h(2\pi)} \widehat{u}_1^3(2\pi, y) = 0.$$

Adding (3.8) and(3.9), and using (2.10) we get

$$2 \int_\Omega \left(\widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial x_1} + \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y} \right) \widehat{u}_2 = \int_{\partial\Omega} \widehat{u}_1 \widehat{u}_2^2 \cos(\vec{n}, x_1) + \int_{\partial\Omega} \widehat{u}_2^3 \cos(\vec{n}, y) =$$

$$= - \int_0^{h(0)} \widehat{u}_1(0, y) \widehat{u}_1^2(0, y) + \int_0^{h(2\pi)} \widehat{u}_1(2\pi, y) \widehat{u}_1^2(2\pi, y) = 0.$$

Theorem 3.1 : For $0 < \varepsilon < 1$ let $\widehat{u}_1, \widehat{u}_2, \widehat{\omega}_3, \widehat{p}$ be the solution of problem $(\widehat{P}^\varepsilon)$. We make the following assumptions :

$$0 < h(x_1) \leq \delta \ , \ 0 \leq x_1 \leq 2\pi \ \text{ with } \delta \text{ small enough , independent of } \varepsilon \tag{3.10}$$

$$R_L = \varepsilon^2 R_c \tag{3.11}$$

$$I.R_L = \varepsilon. \tag{3.12}$$

Then

$$\left\|\varepsilon\frac{\partial\widehat{u}_1}{\partial x_1}\right\|+\left\|\frac{\partial\widehat{u}_1}{\partial y}\right\|+\left\|\varepsilon^2\frac{\partial\widehat{u}_2}{\partial x_1}\right\|+\left\|\varepsilon\frac{\partial\widehat{u}_2}{\partial y}\right\|+\left\|\varepsilon\frac{\partial\widehat{\omega}_3}{\partial x_1}\right\|+\left\|\frac{\partial\widehat{\omega}_3}{\partial y}\right\|+\|\widehat{\omega}_3\|\leq const\,, \quad (3.13)$$

where $\|\cdot\|$ is the usual norm in $L^2(\Omega)$ and *const* depends only on R_L, N and on the $H^1(\Omega)$ norms of J_1, J_2, K.

Proof : We take $\varphi=\widehat{u}_1-\widehat{G}_1$, $\psi=\widehat{u}_2-\widehat{G}_2$ in (2.15) and (2.16) respectively. We get

$$\begin{aligned}&\varepsilon^2\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial x_1}\right)^2+\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2-\int_\Omega\widehat{p}\,\frac{\partial}{\partial x_1}(\widehat{u}_1-\widehat{G}_1)\;=\\&=\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_1}{\partial x_1}\frac{\partial\widehat{G}_1}{\partial x_1}+\int_\Omega\frac{\partial\widehat{u}_1}{\partial y}\frac{\partial\widehat{G}_1}{\partial y}+2N^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1\widehat{G}_1)-\\&-\varepsilon^2R_L\left(\int_\Omega\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}(\widehat{u}_1-\widehat{G}_1)+\int_\Omega\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}(\widehat{u}_1-\widehat{G}_1)\right),\end{aligned}\quad (3.14)$$

and

$$\begin{aligned}&\varepsilon^4\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial x_1}\right)^2+\varepsilon^2\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial y}\right)^2-\int_\Omega\widehat{p}\,\frac{\partial}{\partial y}(\widehat{u}_2-\widehat{G}_2)=\varepsilon^4\int_\Omega\frac{\partial\widehat{u}_2}{\partial x_1}\frac{\partial\widehat{G}_2}{\partial x_1}+\\&\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_2}{\partial y}\frac{\partial\widehat{G}_2}{\partial y}-\;2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{u}_2-\widehat{G}_2)-\\&-\varepsilon^4R_L\left(\int_\Omega\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}(\widehat{u}_2-\widehat{G}_2)+\int_\Omega\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}(\widehat{u}_2-\widehat{G}_2)\right).\end{aligned}\quad (3.15)$$

Adding (3.14) and (3.15) we get

$$\begin{aligned}&\varepsilon^2\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial x_1}\right)^2+\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2+\varepsilon^4\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial x_1}\right)^2+\varepsilon^2\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial y}\right)^2=\\&=\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_1}{\partial x_1}\frac{\partial\widehat{G}_1}{\partial x_1}+\int_\Omega\frac{\partial\widehat{u}_1}{\partial y}\frac{\partial\widehat{G}_1}{\partial y}+\varepsilon^4\int_\Omega\frac{\partial\widehat{u}_2}{\partial x_1}\frac{\partial\widehat{G}_2}{\partial x_1}+\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_2}{\partial y}\frac{\partial\widehat{G}_2}{\partial y}-\\&-2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{u}_2-\widehat{G}_2)\;-\varepsilon^4R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\right)\widehat{u}_2+\\&+\varepsilon^4R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\right)\widehat{G}_2+2N^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1-\widehat{G}_1)\\&-\varepsilon^2R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}\right)\widehat{u}_1+\varepsilon^2R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}\right)\widehat{G}_1.\end{aligned}$$

Using (2.10), lemma 3.2 and $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we obtain

$$\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2+\frac{\varepsilon^4}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial x_1}\right)^2+\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial y}\right)^2 \leq$$

$$\leq 2N^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1-\widehat{G}_1)\ -2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{u}_2-\widehat{G}_2)+\varepsilon^2R_L\int_\Omega\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}\widehat{G}_1+$$

$$+\varepsilon^2R_L\int_\Omega\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}\widehat{G}_1+\varepsilon^4R_L\left(\int_\Omega\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}\widehat{G}_2+\int_\Omega\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\widehat{G}_2\right)+K_1\,, \qquad (3.16)$$

where

$$K_1=\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{G}_1}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{G}_1}{\partial y}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{G}_2}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{G}_2}{\partial y}\right)^2.$$

Taking $\eta=\widehat{\omega}_3-\widehat{F}$ in (2.17) and using the same arguments as before we get

$$\frac{R_L}{2R_c}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial x_1}\right)^2+\frac{R_L}{2R_c\,\varepsilon^2}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial y}\right)^2+2N^2\int_\Omega(\widehat{\omega}_3)^2\leq$$

$$2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_2}{\partial x_1}(\widehat{\omega}_3-\widehat{F})--2N^2\int_\Omega\frac{\partial\widehat{u}_1}{\partial y}(\widehat{\omega}_3-\widehat{F})-$$

$$-I.R_L\int_\Omega\widehat{u}_1\frac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{\omega}_3-\widehat{F})-I.R_L\int_\Omega\widehat{u}_2\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{\omega}_3-\widehat{F})+K_2, \qquad (3.17)$$

where

$$K_2=\frac{R_L}{2R_c}\int_\Omega\left(\frac{\partial\widehat{F}}{\partial x_1}\right)^2+\frac{R_L}{2R_c\,\varepsilon^2}\int_\Omega\left(\frac{\partial\widehat{F}}{\partial y}\right)^2+2N^2\int_\Omega(\widehat{F})^2\,.$$

We add (3.16) and (3.17)

$$\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2+\frac{\varepsilon^4}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial x_1}\right)^2+\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial y}\right)^2+$$

$$+\frac{R_L}{2R_c}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial x_1}\right)^2+\frac{R_L}{2R_c\varepsilon^2}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial y}\right)^2+2N^2\int_\Omega(\widehat{\omega}_3)^2\leq$$

$$-2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{u}_2-\widehat{G}_2)-\varepsilon^2R_L\int_\Omega\left(\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}+\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}\right)\widehat{u}_1+$$

$$+\varepsilon^2R_L\int_\Omega\left(\widehat{u}_2\frac{\partial\widehat{u}_1}{\partial y}+\widehat{u}_1\frac{\partial\widehat{u}_1}{\partial x_1}\right)\widehat{G}_1+2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_2}{\partial x_1}(\widehat{\omega}_3-\widehat{F})+$$

$$\varepsilon^4 R_L\left(\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\right)\widehat{G}_2-\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\right)\widehat{u}_2\right)+$$
$$2N^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1-\widehat{G}_1)-2N^2\int_\Omega\frac{\partial\widehat{u}_1}{\partial y}(\widehat{\omega}_3-\widehat{F})+$$
$$+I.R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{\omega}_3}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{\omega}_3}{\partial y}\right)\widehat{F}-I.R_L\int_\Omega\left(\widehat{u}_1\frac{\partial\widehat{\omega}_3}{\partial x_1}+\widehat{u}_2\frac{\partial\widehat{\omega}_3}{\partial y}\right)\widehat{\omega}_3+K_1+K_2\,. \quad (3.18)$$

Using lemma 3.1 and lemma 3.2, we put

$$A_1 = 2N^2\int_\Omega\tfrac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1-\widehat{G}_1) - 2N^2\varepsilon^2\int_\Omega\tfrac{\partial\widehat{\omega}_3}{\partial x_1}(\widehat{u}_2-\widehat{G}_2),$$

$$A_2 = \varepsilon^2 R_L\int_\Omega\left(\widehat{u}_2\tfrac{\partial\widehat{u}_1}{\partial y}+\widehat{u}_1\tfrac{\partial\widehat{u}_1}{\partial x_1}\right)\widehat{G}_1+\varepsilon^4 R_L\int_\Omega\left(\widehat{u}_1\tfrac{\partial\widehat{u}_2}{\partial x_1}+\widehat{u}_2\tfrac{\partial\widehat{u}_2}{\partial y}\right)\widehat{G}_2,$$

%vspace0.3cm

$$A_3 = 2N^2\varepsilon^2\int_\Omega\tfrac{\partial\widehat{u}_2}{\partial x_1}(\widehat{\omega}_3-\widehat{F}) - 2N^2\int_\Omega\tfrac{\partial\widehat{u}_1}{\partial y}(\widehat{\omega}_3-\widehat{F}),$$

$$A_4 = I.R_L\int_\Omega\left(\widehat{u}_1\tfrac{\partial\widehat{\omega}_3}{\partial x_1}+\widehat{u}_2\tfrac{\partial\widehat{\omega}_3}{\partial y}\right)\widehat{F}.$$

Using (3.11) we can write (3.18) in the following form

$$\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2+\frac{\varepsilon^4}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial x_1}\right)^2+\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{u}_2}{\partial y}\right)^2+$$
$$+\frac{\varepsilon^2}{2}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial x_1}\right)^2+\frac{1}{2}\int_\Omega\left(\frac{\partial\widehat{\omega}_3}{\partial y}\right)^2+2N^2\int_\Omega(\widehat{\omega}_3)^2\le$$
$$A_1+A_2+A_3+A_4+K_1+K_2.$$

Now, we will estimate A_1, A_2, A_3, A_4. Using the usual Poincare inequality :

$$\int_\Omega|v|^2\le\frac{\delta^2}{2}\int_\Omega|\frac{\partial v}{\partial y}|^2, \quad (3.19)$$

for every $v\in H^1(\Omega)$ with zero value on Γ_0 or Γ_1 and using $ab\le\frac{1}{2}\left(a^2+b^2\right)$ in A_1, we get

$$\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial y}(\widehat{u}_1-\widehat{G}_1) \le \left\|\frac{\partial\widehat{\omega}_3}{\partial y}\right\|.\|\widehat{u}_1-\widehat{G}_1\| \le \frac{\delta}{\sqrt{2}}\left\|\frac{\partial\widehat{\omega}_3}{\partial y}\right\|\cdot\left\|\frac{\partial(\widehat{u}_1-\widehat{G}_1)}{\partial y}\right\|$$

$$\le \frac{\delta}{\sqrt{2}}\int_\Omega\left\{\left(\frac{\partial\widehat{\omega}_3}{\partial y}\right)^2+\left(\frac{\partial\widehat{u}_1}{\partial y}\right)^2+\left(\frac{\partial\widehat{G}_1}{\partial y}\right)^2\right\}. \quad (3.20)$$

Similarly, we have

$$\int_\Omega \frac{\partial \widehat{\omega}_3}{\partial y}(\widehat{u}_2 - \widehat{G}_2) \le \frac{\delta}{\sqrt{2}} \int_\Omega \left\{ \left(\frac{\partial \widehat{\omega}_3}{\partial x_1}\right)^2 + \left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2 + \left(\frac{\partial \widehat{G}_2}{\partial y}\right)^2 \right\}. \tag{3.21}$$

From (3.20) and (3.21) we deduce

$$\begin{aligned} A_1 \ &\le \ \frac{2N^2\delta}{\sqrt{2}} \int_\Omega \left\{ \left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + \left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2 + \left(\frac{\partial \widehat{G}_1}{\partial y}\right)^2 \right\} + \\ &+ \frac{2N^2\delta\varepsilon^2}{\sqrt{2}} \int_\Omega \left\{ \left(\frac{\partial \widehat{\omega}_3}{\partial x_1}\right)^2 + \left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2 + \left(\frac{\partial \widehat{G}_2}{\partial y}\right)^2 \right\}. \end{aligned} \tag{3.22}$$

We have

$$\begin{aligned} \int_\Omega \frac{\partial \widehat{u}_2}{\partial x_1}(\widehat{\omega}_3 - \widehat{F}) \le \frac{\delta}{\sqrt{2}} \left\| \varepsilon \frac{\partial \widehat{u}_2}{\partial x_1} \right\| \cdot \left\| \frac{1}{\varepsilon} \frac{\partial \left(\widehat{\omega}_3 - \widehat{F}\right)}{\partial y} \right\| \le \\ \le \frac{\delta}{\sqrt{2}} \int_\Omega \left\{ \varepsilon^2 \left(\frac{\partial \widehat{u}_2}{\partial x_1}\right)^2 + \frac{1}{\varepsilon^2} \left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + \frac{1}{\varepsilon^2} \left(\frac{\partial \widehat{F}}{\partial y}\right)^2 \right\} \end{aligned} \tag{3.23}$$

and

$$\int_\Omega \frac{\partial \widehat{u}_1}{\partial y} \left(\widehat{\omega}_3 - \widehat{F}\right) \le \frac{\delta}{\sqrt{2}} \int_\Omega \left\{ \left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2 + \left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + \left(\frac{\partial \widehat{F}}{\partial y}\right)^2 \right\}. \tag{3.24}$$

Then from (3.23) and (3.24), we get

$$\begin{aligned} A_3 \le \frac{2N^2\delta}{\sqrt{2}} \int_\Omega \left\{ \varepsilon^4 \left(\frac{\partial \widehat{u}_2}{\partial x_1}\right)^2 + \left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + \left(\frac{\partial \widehat{F}}{\partial y}\right)^2 \right\} + \\ \frac{2N^2\delta}{\sqrt{2}} \int_\Omega \left\{ \left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2 + \left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + \left(\frac{\partial \widehat{F}}{\partial y}\right)^2 \right\}. \end{aligned} \tag{3.25}$$

We also have

$$\begin{aligned} A_2 \ &\le \ \varepsilon^2 R_L \left(\int_\Omega \widehat{u}_2 \frac{\partial \widehat{u}_1}{\partial y} \widehat{G}_1 + \int_\Omega \widehat{u}_1 \frac{\partial \widehat{u}_1}{\partial x_1} \widehat{G}_1 \right) + \\ &+ \varepsilon^4 R_L \left(\int_\Omega \widehat{u}_1 \frac{\partial \widehat{u}_2}{\partial x_1} \widehat{G}_2 + \int_\Omega \widehat{u}_2 \frac{\partial \widehat{u}_2}{\partial y} \widehat{G}_2 \right). \end{aligned}$$

Since $\widehat{G}=(\widehat{G}_1,\widehat{G}_2)\in(L^\infty(\Omega))^2$, using (3.19) we obtain

$$\begin{aligned}\int_\Omega \widehat{u}_2\frac{\partial \widehat{u}_1}{\partial y}\widehat{G}_1+\int_\Omega \widehat{u}_1\frac{\partial \widehat{u}_1}{\partial x_1}\widehat{G}_1 &\le M\left(\|\widehat{u}_2\|\left\|\frac{\partial \widehat{u}_1}{\partial y}\right\|+\|\widehat{u}_1\|\left\|\frac{\partial \widehat{u}_1}{\partial x_1}\right\|\right)\le\\ &\le \frac{\delta M}{\sqrt{2}}\left(\left\|\frac{\partial \widehat{u}_2}{\partial y}\right\|\left\|\frac{\partial \widehat{u}_1}{\partial y}\right\|+\left\|\frac{\partial \widehat{u}_1}{\partial x_1}\right\|\left\|\frac{\partial \widehat{u}_1}{\partial x_1}\right\|\right)\le\\ &\le\frac{\delta M}{2\sqrt{2}}\int_\Omega\left\{\left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2+\left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2+\left(\frac{\partial \widehat{u}_1}{\partial x_1}\right)^2\right\},\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}\int_\Omega \widehat{u}_1\frac{\partial \widehat{u}_2}{\partial x_1}\widehat{G}_2+\int_\Omega \widehat{u}_2\frac{\partial \widehat{u}_2}{\partial y}\widehat{G}_2 &\le M\left(\left\|\frac{1}{\varepsilon}\widehat{u}_1\right\|\left\|\varepsilon\frac{\partial \widehat{u}_2}{\partial x_1}\right\|+\|\widehat{u}_2\|\left\|\frac{\partial \widehat{u}_2}{\partial y}\right\|\right)\le\\ &\le \frac{\delta M}{\sqrt{2}}\left(\left\|\frac{1}{\varepsilon}\frac{\partial \widehat{u}_1}{\partial y}\right\|\left\|\varepsilon\frac{\partial \widehat{u}_2}{\partial x_1}\right\|+\left\|\frac{\partial \widehat{u}_2}{\partial y}\right\|^2\right)\le\\ &\le\frac{\delta M}{2\sqrt{2}}\int_\Omega\left\{\frac{1}{\varepsilon^2}\left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2+\varepsilon^2\left(\frac{\partial \widehat{u}_2}{\partial x_1}\right)^2+\left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2\right\}.\end{aligned}\tag{3.27}$$

From (3.26) and (3.27) we obtain

$$\begin{aligned}A_2 &\le \frac{R_L\delta M\varepsilon^2}{2\sqrt{2}}\int_\Omega\left\{\left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2+\left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2+\left(\frac{\partial \widehat{u}_1}{\partial x_1}\right)^2\right\}+\\ &+\frac{R_L\delta M\varepsilon^2}{2\sqrt{2}}\int_\Omega\left\{\left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2+\varepsilon^4\left(\frac{\partial \widehat{u}_2}{\partial x_1}\right)^2+\varepsilon^2\left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2\right\}.\end{aligned}\tag{3.28}$$

And as

$$A_4=-I.R_L\int_\Omega \widehat{u}_1\frac{\partial \widehat{\omega}_3}{\partial x_1}\widehat{F}-I.R_L\int_\Omega \widehat{u}_2\frac{\partial \widehat{\omega}_3}{\partial y}\widehat{F}.$$

Also

$$\begin{aligned}I.R_L\int_\Omega \widehat{u}_2\frac{\partial \widehat{\omega}_3}{\partial y}\widehat{F}\le\int_\Omega |I.R_LM\widehat{u}_2|\left|\frac{\partial \widehat{\omega}_3}{\partial y}\right| &\le \frac{1}{16}\int_\Omega\left|\frac{\partial \widehat{\omega}_3}{\partial y}\right|^2+\\ +16(I.R_LM)^2\int_\Omega|\widehat{u}_2|^2\le\frac{1}{16}\int_\Omega\left|\frac{\partial \widehat{\omega}_3}{\partial y}\right|^2+\frac{16(I.R_LM\delta)^2}{2}\int_\Omega\left|\frac{\partial \widehat{u}_2}{\partial y}\right|^2,\end{aligned}$$

and

$$\begin{aligned}I.R_L\int_\Omega \widehat{u}_1\frac{\partial \widehat{\omega}_3}{\partial x_1}\widehat{F}\le\int_\Omega\left|\frac{I\,R_LM}{\varepsilon}\widehat{u}_1\right|\left|\varepsilon\frac{\partial \widehat{\omega}_3}{\partial x_1}\right| &\le \frac{\varepsilon^2}{16}\int_\Omega\left|\frac{\partial \widehat{\omega}_3}{\partial x_1}\right|^2+\\ +\frac{16(I.R_LM)^2}{\varepsilon^2}\int_\Omega|\widehat{u}_1|^2\le\frac{\varepsilon^2}{16}\int_\Omega\left|\frac{\partial \widehat{\omega}_3}{\partial x_1}\right|^2+\frac{16(I.R_LM\delta)^2}{2\varepsilon^2}\int_\Omega\left|\frac{\partial \widehat{u}_1}{\partial y}\right|^2.\end{aligned}\tag{3.29}$$

From (3.29) and (3.29) and using (3.12) we get

$$A_4 \leq \frac{1}{16}\int_\Omega \left|\frac{\partial \widehat{\omega}_3}{\partial y}\right|^2 + \frac{16(I.R_L M\delta)^2}{2}\int_\Omega \left|\frac{\partial \widehat{u}_2}{\partial y}\right|^2 + \frac{\varepsilon^2}{16}\int_\Omega \left|\frac{\partial \widehat{\omega}_3}{\partial x_1}\right|^2$$
$$+\frac{16(I.R_L M\delta)^2}{2\varepsilon^2}\int_\Omega \left|\frac{\partial \widehat{u}_1}{\partial y}\right|^2 = \frac{1}{16}\int_\Omega \left|\frac{\partial \widehat{\omega}_3}{\partial y}\right|^2 +$$
$$+8(M\delta)^2\varepsilon^2\int_\Omega \left|\frac{\partial \widehat{u}_2}{\partial y}\right|^2 + \frac{\varepsilon^2}{16}\int_\Omega \left|\frac{\partial \widehat{\omega}_3}{\partial x_1}\right|^2 + 8(M\delta)^2\int_\Omega \left|\frac{\partial \widehat{u}_1}{\partial y}\right|^2. \tag{3.30}$$

From (3.22),(3.25),(3.28) and (3.30) we deduce

$$\alpha_1\int_\Omega\left(\frac{\partial \widehat{u}_1}{\partial x_1}\right)^2 + \alpha_2\int_\Omega\left(\frac{\partial \widehat{u}_1}{\partial y}\right)^2 + \alpha_3\int_\Omega\left(\frac{\partial \widehat{u}_2}{\partial x_1}\right)^2 + \alpha_4\int_\Omega\left(\frac{\partial \widehat{u}_2}{\partial y}\right)^2 +$$
$$\alpha_5\int_\Omega\left(\frac{\partial \widehat{\omega}_3}{\partial x_1}\right)^2 + \alpha_6\int_\Omega\left(\frac{\partial \widehat{\omega}_3}{\partial y}\right)^2 + + \alpha_7\int_\Omega(\widehat{\omega}_3)^2 \leq K$$

where

$$\alpha_1 = \frac{\varepsilon^2}{2}\left(1-\frac{R_L\delta M}{\sqrt{2}}\right), \quad \alpha_2 = \frac{1}{2} - \left(\frac{4N^2\delta}{\sqrt{2}} + \frac{8(I\,R_L M\delta)^2}{\varepsilon^2} + \frac{R_L\delta M\varepsilon^2}{\sqrt{2}}\right),$$
$$\alpha_3 = \frac{\varepsilon^4}{4}\left(1-\frac{8N^2\delta}{\sqrt{2}} - \frac{4R_L\delta M\varepsilon^2}{2\sqrt{2}}\right),$$
$$\alpha_4 = \frac{\varepsilon^2}{2}\left(1-\frac{4N^2\delta}{\sqrt{2}} - \frac{R_L\delta M}{2} - 16(I\,R_L M\delta)^2\right),$$

$$\alpha_5 = \frac{R_L}{2R_c} - \left(\frac{2N^2\delta\varepsilon^2}{\sqrt{2}} + \frac{\varepsilon^2}{16}\right) \alpha_6 = \frac{R_L}{2R_c\varepsilon^2} - \left(\frac{6N^2\delta}{\sqrt{2}} + \frac{1}{16}\right), \; \alpha_7 = 2N^2,$$

and

$$K = \frac{1}{2}\int_\Omega\left(\frac{\partial \widehat{G}_1}{\partial x_1}\right)^2 + \frac{1}{2}\int_\Omega\left(\frac{\partial \widehat{G}_1}{\partial y}\right)^2 + \frac{1}{2}\int_\Omega\left(\frac{\partial \widehat{G}_2}{\partial x_1}\right)^2 + \frac{1}{2}\int_\Omega\left(\frac{\partial \widehat{G}_2}{\partial y}\right)^2$$
$$+\frac{R_L}{2R_c}\int_\Omega\left(\frac{\partial \widehat{F}}{\partial x_1}\right)^2 +$$
$$+\frac{R_L}{2R_c\,\varepsilon^2}\int_\Omega\left(\frac{\partial \widehat{F}}{\partial y}\right)^2 + 2N^2\int_\Omega(\widehat{F})^2 + \frac{2N^2\delta}{\sqrt{2}}\int_\Omega\left(\frac{\partial \widehat{G}_1}{\partial y}\right)^2 +$$

$$+\frac{2N^2\delta\varepsilon^2}{\sqrt{2}}\int_\Omega\left(\frac{\partial\widehat{G}_2}{\partial y}\right)^2+\frac{4N^2\delta}{\sqrt{2}}\int_\Omega\left(\frac{\partial\widehat{F}}{\partial y}\right)^2.$$

Finally, for δ small enough, (3.13) follows.

Now, we estimate the pressure uniformly with respect to ε, using the following result:
Lemma 3.3 [1]: $\forall\varphi\in H^1(\Omega)$, $\varphi=k$ on $\partial\Omega$, we have

$$|\varphi|_{L^4(\Omega)}\ \le\ C(\Omega,k)\ (\ \left\|\frac{\partial\varphi}{\partial x_1}\right\|^{\frac14}\left\|\frac{\partial\varphi}{\partial y}\right\|^{\frac34}+\left\|\frac{\partial\varphi}{\partial y}\right\|^{\frac12}) \tag{3.31}$$

Lemma 3.4 : Under the assumptions of Theorem 3.1 we have the following estimates:

$$\left\|\frac{\partial\widehat{p}}{\partial x_1}\right\|_{H^{-1}(\Omega)}\le const, \tag{3.32}$$

$$\left\|\frac{\partial\widehat{p}}{\partial y}\right\|_{H^{-1}(\Omega)}\le \varepsilon.const \tag{3.33}$$

where *const* depends only on $H^1(\Omega)$ norms of J_1, J_2 and K.

Proof : From the identity (2.16), we have

$$-\int_\Omega\widehat{p}\frac{\partial\psi}{\partial y}\ =\ <\frac{\partial\widehat{p}}{\partial y}\,,\,\psi>_{H^{-1},H_0^1}=-\varepsilon^4\int_\Omega\frac{\partial\widehat{u}_2}{\partial x_1}\frac{\partial\psi}{\partial x_1}-\varepsilon^2\int_\Omega\frac{\partial\widehat{u}_2}{\partial y}\frac{\partial\psi}{\partial y}-$$

$$2N^2\varepsilon^2\int_\Omega\frac{\partial\widehat{\omega}_3}{\partial x_1}\psi\ -\varepsilon^4R_L\int_\Omega\widehat{u}_1\frac{\partial\widehat{u}_2}{\partial x_1}\psi-\varepsilon^4R_L\int_\Omega\widehat{u}_2\frac{\partial\widehat{u}_2}{\partial y}\psi, \tag{3.34}$$

whence

$$\left|<\frac{\partial\widehat{p}}{\partial y},\psi>\right|\ \le\ \varepsilon^4\left\|\frac{\partial\widehat{u}_2}{\partial x_1}\right\|\ \left\|\frac{\partial\psi}{\partial x_1}\right\|+\varepsilon^2\left\|\frac{\partial\widehat{u}_2}{\partial y}\right\|\ \left\|\frac{\partial\psi}{\partial y}\right\|+2N^2\varepsilon^2\left\|\frac{\partial\widehat{\omega}_3}{\partial x_1}\right\|\ \|\psi\|+$$

$$+\ \varepsilon^4R_L\|\widehat{u}_1\|_{L^4(\Omega)}\ \left\|\frac{\partial\widehat{u}_2}{\partial x_1}\right\|\ \|\psi\|_{L^4(\Omega)}+\varepsilon^4R_L\|\widehat{u}_2\|_{L^4(\Omega)}\ \left\|\frac{\partial\widehat{u}_2}{\partial y}\right\|\ \|\psi\|_{L^4(\Omega)}.$$

Using Lemma 3.3, we get

$$\left|<\frac{\partial\widehat{p}}{\partial y},\psi>\right|\le C'\ \left[\varepsilon^2\ \left\|\varepsilon^2\frac{\partial\widehat{u}_2}{\partial x_1}\right\|+\varepsilon\left\|\varepsilon\frac{\partial\widehat{u}_2}{\partial y}\right\|+\varepsilon N^2\left\|\varepsilon\frac{\partial\widehat{\omega}_3}{\partial x_1}\right\|\ \right]\ \|\psi\|_{H_0^1(\Omega)}+$$

$$+\ C'\ \left[\varepsilon^4R_L\left(\left\|\frac{\partial\widehat{u}_1}{\partial x_1}\right\|^{\frac14}\ \left\|\frac{\partial\widehat{u}_1}{\partial y}\right\|^{\frac34}+\left\|\frac{\partial\widehat{u}_1}{\partial y}\right\|^{\frac12}\right)\left\|\frac{\partial\widehat{u}_2}{\partial x_1}\right\|\ \right]\ \|\psi\|_{H_0^1(\Omega)}\ +$$

$$+\ C'\ \left[\varepsilon^4R_L\left(\left\|\frac{\partial\widehat{u}_2}{\partial x_1}\right\|^{\frac14}\left\|\frac{\partial\widehat{u}_2}{\partial y}\right\|^{\frac34}+\left\|\frac{\partial\widehat{u}_2}{\partial y}\right\|^{\frac12}\right)\left\|\frac{\partial\widehat{u}_2}{\partial y}\right\|\right]\ \|\psi\|_{H_0^1(\Omega)},$$

where C' is a positive constant which depend only on Ω, J_1, J_2, K, R_L and N. Using (3.13) we deduce

$$< \frac{\partial \widehat{p}}{\partial y}, \psi > \; \leq \; C' \left(\varepsilon + \varepsilon^2 + N^2\varepsilon^2 + R_L \left(\varepsilon^{\frac{7}{4}} + \varepsilon^2 + \varepsilon^{\frac{5}{4}} + \varepsilon^{\frac{5}{2}}\right)\right) \|\psi\|_{H_0^1(\Omega)}.$$

Then

$$\left\|\frac{\partial \widehat{p}}{\partial y}\right\|_{H^{-1}(\Omega)} \leq C\varepsilon,$$

where C is a positive constant which depends on Ω, J_1, J_2, K, R_L and N.
Using the same argument from the identity (2.15), we deduce (3.32).
Having estimates (3.13),(3.32) and (3.33) we can select a subsequence $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{\omega}_3^\varepsilon, \widehat{p}^\varepsilon)$ of solutions to problems $\left(\widehat{P}^\varepsilon\right)$ which converges to some limit functions $u_1^*, u_2^*, \omega_3^*, p^*$ in suitable topologies. Let

$$V_y = \left\{ v \in L^2(\Omega) : \frac{\partial v}{\partial y} \in L^2(\Omega) \right\}$$

and let $\|v\| + \left\|\frac{\partial v}{\partial y}\right\|$ be the norm in V_y.

Theorem 3.2 : Let the assumptions of Theorems 2.1 and 2.2 hold, then there exists functions $u_1^*, \omega_3^* \in V_y$ and $p^* \in L_0^2(\Omega)$ such that (for a subsequence)

$$\widehat{u}_1^\varepsilon \to u_1^* \quad \text{weakly in } V_y \tag{3.35}$$

$$\widehat{\omega}_3^\varepsilon \to \omega_3^* \quad \text{weakly in } V_y \tag{3.36}$$

$$\widehat{p}^\varepsilon \to p^* \quad \text{weakly in } L_0^2(\Omega). \tag{3.37}$$

Moreover if $\nu_\gamma R_c$ is equivalent to ε^{-2}, then there exists N_0 such that $N_0 = \lim_{\varepsilon\to 0} N$, and the following relations hold :

$$\frac{\partial p^*}{\partial x_1} = \frac{\partial^2 u_1^*}{\partial y^2} + 2N_0^2 \frac{\partial \omega_3^*}{\partial y} \qquad \text{in } H^{-1}(\Omega)\,, \tag{3.38}$$

$$\frac{\partial p^*}{\partial y} = 0 \qquad \text{a.e. in } \Omega \tag{3.39}$$

$$\frac{\partial^2 \omega^*}{\partial y^2} = 2N_0^2 \left(2\omega_3^* + \frac{\partial u_1^*}{\partial y}\right) \quad \text{in } H^{-1}(\Omega) \tag{3.40}$$

Proof : Identities (3.35) and (3.36) follow immediately from estimate (3.13). As $\widehat{p}^\varepsilon$ is in $L_0^2(\Omega)$ and is uniformly bounded in $H^{-1}(\Omega)$, we have [14]

$$\|\widehat{p}^\varepsilon\| \leq C \, \|\widehat{p}^\varepsilon\|_{H^{-1}(\Omega)} \leq const\,,$$

so that there exists a subsequence $(\widehat{p}^\varepsilon)$ converging weakly in $L^2(\Omega)$ to some p^*, and

$$\|p^*\| \;\leq\; lim\,inf_{\varepsilon\to 0} \|\widehat{p}^\varepsilon\| \leq const.$$

Moreover, $p^* \in L_0^2(\Omega)$ as $L_0^2(\Omega)$ is weakly closed in $L^2(\Omega)$.
Using (3.35)-(3.37) we pass to the limit as $\varepsilon \to 0$ in (2.15)-(2.17), where R_L,R_c and I are as in (3.11) and (3.12), and obtain for all functions $\varphi, \psi, \eta \in H_0^1(\Omega)$:

$$\int_\Omega \frac{\partial u_1^*}{\partial y}\frac{\partial \varphi}{\partial y} - \int_\Omega p^* \frac{\partial \varphi}{\partial x_1} = 2N_0^2 \int_\Omega \frac{\partial \omega_3^*}{\partial y}\varphi, \tag{3.41}$$

$$\int_\Omega -p^* \frac{\partial \psi}{\partial y} = 0, \tag{3.42}$$

$$\int_\Omega \frac{\partial \omega_3^*}{\partial y}\frac{\partial \eta}{\partial y} + 4N_0^2 \int_\Omega \omega_3^* \eta = -2N_0^2 \int_\Omega \frac{\partial u_1^*}{\partial y}\eta. \tag{3.43}$$

This proves (3.38)-(3.40).
The study of the limit equation (3.38)-(3.40), can be found in [3], where a higher regularity of u_1^*, ω_3^*, p^* is proved then this system was exlicitly solved in u_1^* and ω_3^*, to obtain the generalized Reynolds equation (1.14).

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More Lyapunov functions for the Navier-Stokes equations

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0 Introduction

We are interested in solving the three-dimensional incompressible Navier-Stokes system in the whole space, say

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u &= 0, \\ u(x,0) &= u_0(x),\ x \in \mathbb{R}^3,\ t \geq 0. \end{cases} \tag{0.1}$$

It is well-known that local (or global under smallness assumptions on u_0) well-posedness holds in various functional spaces ([19, 18, 3, 22]). A classical example is provided by the space of continuous in time solutions taking values in the Sobolev space $\dot{H}^{\frac{1}{2}}$, say $u \in C_t(\dot{H}^{\frac{1}{2}})$. Moreover, we have at our disposal a persistence result, namely: if the initial data u_0 is not only small in $\dot{H}^{\frac{1}{2}}$, but also belongs to H^1, then the above mentioned solution u is globally continuous in time with values in H^1. To prove such a result it is enough to show that the H^1 norm of the solution is a Lyapunov function, which means a decreasing in time one. More accurately, in the celebrated paper by T. Kato and H. Fujita [19] the following inequality is proven :

$$\frac{d}{dt}\|\nabla u\|_2^2 \leq -2\|\nabla u\|_2^2(1 - C\|u\|_{\dot{H}^{\frac{1}{2}}}), \tag{0.2}$$

that immediately provides the announced decreasing in time of the homogeneous norm $||u||_{\dot{H}^1}$, as long as $||u||_{\dot{H}^{\frac{1}{2}}}$ is small enough. On the other hand the L^2 norm of the solution u decreases as well, which allows to conclude. T. Kato [20] extended this kind of result, actually proving the following inequality :

$$\frac{d}{dt}||J^s u||_p^p \lesssim -(1 - C||u||_3)\, Q_p(J^s u), \tag{0.3}$$

where $J = (I - \Delta)^{\frac{1}{2}}$, $s \geq 0$, $2 \leq p < \infty$ and Q_p is a quantity depending on u and its derivatives. In other words, a new family of Lyapunov functions is derived, and they do not necessarily arise from an energy norm.

The aim of this paper is to adapt Kato's result to the more general Besov spaces that appear since 1995 ([3]) to be better suited for the Navier-Stokes equations. The importance of this result in connection with the stability theory is extensively discussed in [5]. Generally speaking, stability theory enables one to derive from first principles the critical values which separate the different regimes of flows as well as the types of the fluid motions is these different regimes. The time evolution of Lyapunov functions is a suitable property to describe the dissipativity of a system and the stability of its equilibrium solutions. For the Boltzmann equation, the so called H-function (that is a decreasing in time one) plays a key role in the discussion of the change of rate of entropy for a rarefied gas.

For the Navier-Stokes equations, the decreasing in time of the L^2 norm is a well-known property, whose importance is capital for the analysis of Leray's weak solutions. In this case, the usual procedure is to argue on the basis of continuous dependence properties of weak solutions derivable by energy methods and Gronwall type inequalities. In other words, weak flows satisfying the Navier-Stokes equations have the property that, when the viscosity of the fluid is greater than a critical value, all solutions of the initial Cauchy problem tend monotonically to a single flow (the basic one) and the energy of any disturbance of this flow decays from the initial instant.

Very recently, following a tradition inaugurated by T. Kato and H. Fujita [15, 19, 28, 29, 16, 18], a lot of attention has been devoted to the study of the existence and uniqueness of global strong regular solutions not necessarily having a finite energy [3, 4, 5, 6, 7, 8, 14, 21, 24]. Less is known with regard to the existence of Lyapunov functions associated to these strong regular solutions, a part from the above mentioned result obtained in [20] by T. Kato.

As announced, we will concentrate our attention to the Besov frame, even if a careful analysis may allow to consider other "esoteric" spaces such that, for instance, the Morrey-Campanato, Lorentz, BMO and Triebel ones. Our motivation is clearly explained in D. D. Joseph [17] "It is sometimes possible to find positive definite functionals of the disturbance of a basic flow, other than energy, which decrease on the solutions when the viscosity is larger than a critical value. Such functionals, which may be called generalized energy functionals of the Lyapunov type, are of interest because they can lead to a larger interval of viscosities on which the global stability of the basic flow can be guaranteed."

Another motivation for looking at Lyapunov functions in different functional spaces for Navier-Stokes is that they allow to get uniform estimates for the solutions $u(t)$ of the type $\|u(t)\| \leq \|u_0\|$. Now, it is well-known that such an estimate, via a bootstrap argument, allows to pass from a *local* in time solution to a *global* one. This is for instance the case for the Sobolev space $\dot{H}^1(\mathbb{R}^3)$, for which, as we have already recalled, a Lyapunov function exists if the norm of the initial data is sufficiently small in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ [20, 19]. Now, in the case of the *super-critical* space $L^p(\mathbb{R}^3)$, for which a local in time solution is known to exists, it would be of paramount interest to dispose as well of a Lyapunov function, for, finally, this would give a global existence result.

Before proceeding, we would like to add here some remarks on the original proof of T. Kato [20]. The key estimate that guarantees the monotonically decreasing property of the $L^3(\mathbb{R}^3)$ norm is given by eq. (0.3), say

$$\frac{d}{dt}\|u(t)\|_3^3 \leq -C(\nu - K\|u(t)\|_3)Q(u(t)), \tag{0.4}$$

where C and K are two positive constants and $Q(u(t))$ is defined by

$$Q(u(t)) = \int_{\mathbb{R}^3} |u(t)|^3 |\nabla u(t)|^2 dx. \tag{0.5}$$

Of course, $Q(u(t))$ is a well-defined function of t if $u(t)$ is the solution given by T. Kato's celebrated paper [18, 14]. In fact, the solution $u(t)$ not only belongs to $L^\infty(\mathbb{R}^3)$, but also [18] $t\nabla u(t) \in L^\infty(\mathbb{R}^3)$, so that $\forall t > 0$, $|u(t)||\nabla u(t)|^2 \in L^1(\mathbb{R}^3)$. What it less easy to ensure is that $Q(u(t))$ is an $L^1_{loc}(\mathbb{R})$ function of t. And this is evidently of paramount importance when deducing the decreasing property from the estimate (0.4). This is why T. Kato has to impose the additional condition

$$\nabla v(t) \in L^1_{loc}((0,T; L^3(\mathbb{R}^3)). \tag{0.6}$$

We will see how to avoid this extra condition (that does not follow trivially from any well-known estimates verified by the solution $u \in C_t(L^3)$, even if it may be derived from the estimates proved in [6])).

Finally, let us recall here the definition of the Besov spaces that will be used in the following pages. The reader will find a more detailed analysis in the classical books [1, 23, 13] :

DEFINITION 1
Let $\psi \in S$ such that $\hat{\psi} \equiv 1$for $1 < \xi < 2$ and vanishing outside the ball $\frac{1}{2} < \xi < \frac{5}{2}$. We define $\Delta_j f = 2^{nj}\psi(2^j\cdot) * f$. Then, a tempered distribution f belongs to the homogeneous Besov space $\dot{B}^{s,q}_p(\mathbb{R}^n)$ if $\sum_j \Delta_j f$ converges to f in "some sense" and if

$$2^{js}\|\Delta_j f\|_p = \varepsilon_j \in l^q. \tag{0.7}$$

The last quantity serves to define the norm, say $\|f\|_{\dot{B}^{s,q}_p} = (\sum_j \varepsilon_j^q)^{\frac{1}{q}}$.

1 The theorem and its proof

Our main result will be

THEOREM 1
Let u_0 be an initial data in $\dot{B}_p^{s,q}$, with $s = \frac{3}{p} - 1$, $p, q \geq 2$, such that

$$\frac{3}{p} + \frac{2}{q} > 1. \tag{1.1}$$

Then, there exists a constant $C(p,q)$ such that if a solution u of (0.1) verifies the estimate

$$\sup_t \|u(x,t)\|_{\dot{B}_\infty^{-1,\infty}} < C(p,q), \tag{1.2}$$

the norm $\|u(x,t)\|_{\dot{B}_p^{s,q}}$ is a Lyapunov function, that is a decreasing in time one.

Before proceeding with the proof we would like to make some remarks: typical spaces satisfying the hypothesis of the theorem are $\dot{H}^{\frac{1}{2}}$, $\dot{B}_3^{0,3}$ (which is slightly bigger than L^3), and $\dot{B}_4^{-\frac{1}{4},4}$. On the other hand, the existence of solutions to the Navier-Stokes equations in these spaces is ensured by the recent result of Koch and Tataru in BMO^{-1} [19] ; by using the trivial estimate $\|u(x,t)\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \|u(x,t)\|_{BMO^{-1}}$, eq. (1.2) can be seen as an extra smallness condition on the initial data to the necessary assumption needed to prove the existence. We also remark that the constant $C(p,q)$ tends to zero when one approaches the equality sign in condition (1.1). Finally, the proof of the theorem is based in a crucial way on the use of the dyadic norms as recalled in the introduction.

The main idea of the proof is simple and is inspired by Kato's approach introduced in [20]. In fact, we will derive a differential inequality similar to eq. (0.3) but for the norms constructed from the dyadic blocks $\Delta_j u$. In order to proceed, we apply the operator of localization in frequency Δ_j to the equation (0.1), then we multiply by $|\Delta_j u|^{p-2}\Delta_j u$ and get :

$$\frac{d}{dt}\|\Delta_j u\|_p^p + c_p \int |\Delta_j u|^{p-2}|\nabla \Delta_j u|^2 dx \lesssim \int |\Delta_j u|^{p-1}|\Delta_j \mathbb{P}\nabla(u \otimes u)| dx, \tag{1.3}$$

where we integrated by parts the term containing the Laplacian, $\mathbb{P}$ being the projection onto the divergence free vector fields.

Let us recall here that, when $p = 2$, the previous estimate is nothing more than the classical energy estimate allowing the construction of solutions in $\dot{H}^{\frac{1}{2}}$ as in [8, 9]. However, when $p \neq 2$, the second term appearing in the l.h.s. of (1.3) is not easy to be dealt with. The frequency localization helps through the following lemma of Poincaré type [25]:

LEMMA 1

Let f be a function in the Schwartz class whose Fourier transform is supported outside the ball $B(0,1)$.Then, for any real $p \geq 2$,

$$\int |f|^p \leq C_p \int |\nabla f|^2 |f|^{p-2}. \tag{1.4}$$

The reader is referred to [25] for a more general version of the lemma as well as a its proof which is based on integration by parts in each directions x_i, after spitting the Fourier space into conic sectors around ξ_i. This lemma has to be seen as a generalization of a previous result by R. Danchin [11], where the same lemma is proved under some additional restrictions ($p \in 2\mathbb{N}$ as well as a restriction of the localization of the support of $\hat{f}$), and was also applied in a recent work by the same author [12].

Applying the previous lemma we find

$$\frac{d}{dt}\|\Delta_j u\|_p^p + c_p 2^{2j}\|\Delta_j u\|_p^p \lesssim \int |\Delta_j u|^{p-1}|\Delta_j \mathbb{P}\nabla(u\otimes u)|dx. \tag{1.5}$$

We are now left with the estimate of the right hand side. To this end, let us make use of J.M. Bony's paraproduct techniques ([2]) : we denote by $S_j = \sum_{k<j}\Delta_k$ so that $\Delta_j(u^2)$ is the sum of two types of terms, $S_{j-2}u\Delta_j u$ and $\Delta_j(\sum_{k>j}(\Delta_k u)^2)$ (the ones with $\Delta_k u \Delta_{k+1} u$ can be treated in the same way). For the first term we get :

$$\begin{aligned}\int |\Delta_j u|^{p-1}|\mathbb{P}\nabla(S_{j-2}u\Delta_j u)|dx &\lesssim \|\Delta_j u\|_p^{p-1}\|\mathbb{P}\nabla(S_{j-2}u\Delta_j u)\|_p \\ \lesssim \|\Delta_j u\|_p^{p-1}2^j\|S_{j-2}u\Delta_j u)\|_p &\lesssim 2^{2j}\|\Delta_j u\|_p^p \sup_t(2^{-j}\|S_{j-2}u\|_\infty),\end{aligned}$$

by using the continuity of $\mathbb{P}$ in L^p as well as Bernstein inequalities.

Now, we recall that

$$\|u(x,t)\|_{\dot{B}_\infty^{-1,\infty}} \approx \sup_j 2^{-j}\|S_j u\|_\infty \approx \sup_j 2^{-j}\|\Delta_j u\|_\infty, \tag{1.6}$$

therefore under the hypothesis $\sup_t \|u(x,t)\|_{\dot{B}_\infty^{-1,\infty}} < \varepsilon_0$, we get

$$\int |\Delta_j u|^{p-1}|\mathbb{P}\nabla(S_{j-2}u\Delta_j u)|dx \lesssim \varepsilon_0 2^{2j}\|\Delta_j u\|_p^p, \tag{1.7}$$

and we can include this term in the second one appearing in the l.h.s. of (1.5).

Let us consider now the diagonal terms in the paraproduct decomposition and proceed as before :

$$\begin{aligned}&\int |\Delta_j u|^{p-1}|\mathbb{P}\nabla(\Delta_j(\sum_{k>j}(\Delta_k u)^2))|dx \\ &\lesssim \|\Delta_j u\|_\infty \|\Delta_j u\|_p^{p-2}\|\mathbb{P}\nabla(\Delta_j(\sum_{k>j}(\Delta_k u)^2))\|_p \\ &\lesssim \|\Delta_j u\|_\infty \|\Delta_j u\|_p^{p-2}2^j\sum_{k>j}\|\Delta_k u\|_p^2 \lesssim 2^{2j}\varepsilon_0\|\Delta_j u\|_p^{p-2}\sum_{k>j}\|\Delta_k u\|_p^2.\end{aligned}$$

We have thus obtained, after obvious simplifications, the following inequality,

$$\frac{d}{dt}\|\Delta_j u\|_p^2 + c_p 2^{2j}\|\Delta_j u\|_p^2 \lesssim 2^{2j}\varepsilon_0 \sum_{k>j}\|\Delta_k u\|_p^2, \tag{1.8}$$

which, after multiplication by $2^{jqs}\|\Delta_j u\|_p^{q-2}$, becomes (here $r = 2/q + s > 0$)

$$\frac{d}{dt}(2^{jqs}\|\Delta_j u\|_p^q) + c_p 2^{rqj}\|\Delta_j u\|_p^q \lesssim \varepsilon_0 2^{(q-2)rj}\|\Delta_j u\|_p^{q-2} 2^{2rj}\sum_{k>j}\|\Delta_k u\|_p^2. \tag{1.9}$$

Let us denote by $f_j = 2^{js}\|\Delta_j u\|_p$ et $g_j = 2^{jr}\|\Delta_j u\|_p$, the inequality becomes (after summing over j)

$$\frac{d}{dt}(\sum_j f_j^q) + c_p \sum_j g_j^q \lesssim \sum_{k>j} \varepsilon_0 g_j^{q-2} 2^{2rj} 2^{-2rk} g_k^2. \tag{1.10}$$

Remark that, k being fixed, $\sum_{k>j} g_j^{q-2} 2^{2rj} = 2^{2rk}\mu_k^{q-2}$ and $\sum_k \mu_k^q \lesssim \sum_j g_j^q$ (as a convolution between $l^1 - l^{\frac{q}{q-2}}$), and get

$$\frac{d}{dt}(\sum_j f_j^q) + c_p \sum_j g_j^q \lesssim \sum_k \varepsilon_0 \mu_k^{q-2} g_k^2, \tag{1.11}$$

and by using Hölder's inequality we finally obtain

$$\frac{d}{dt}(\sum_j f_j^q) + c_{p,q}(1-\varepsilon_0)\sum_j g_j^q \lesssim 0. \tag{1.12}$$

We observe now that a solution u associated to an initial condition $u_0 \in \dot{B}_p^{s,q}$ verifies the following property ([10])

$$2^{qrj}\int_0^t \|\Delta_j u\|_p^q ds = \eta_j(t) \text{ et } \sum_j \eta_j \lesssim \|u_0\|^q_{\dot{B}_p^{s,q}}. \tag{1.13}$$

Besides, such quantities can be used to construct directly a solution to the Navier-stokes equations via the fixed point theorem ([9]). As far as we are concerned, we can integrate (1.12) to obtain

$$\|u(x,t)\|^q_{\dot{B}_p^{s,q}} + c_{p,q}(1-\varepsilon_0)\int_0^t \|u(x,s)\|^q_{\dot{B}_p^{r,q}} ds \le \|u_0\|^q_{\dot{B}_p^{s,q}}, \tag{1.14}$$

which is the announced decreasing in time property for the Besov norm. This is enough to conclude the proof of the theorem.

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On the Nonlinear Stability of the Magnetic Bénard Problem

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1 Introduction

The magnetic Bénard problem concerns the onset of convection in a horizontal layer of a heavy, incompressible, viscous, electrically conducting fluid, heated from below, when a vertical constant magnetic field is imposed and the Boussinesq approximation is adopted. In this case, the rest state is a possible solution for all values of the temperature gradient β and of the intensity of the magnetic field H; however, such basic solution is observable only for suitable combinations of the values β and H. While he linear stability studies the limit relation between such values, that is, the critical linear Rayleigh number $\mathcal{R}_L$, as function of the Hartman number Q_L, above which the convection sets in, the nonlinear stability looks for a corresponding limit relation between critical nonlinear Rayleigh number $\mathcal{R}_n$ and the Hartman number Q_n, below which the rest is nonlinearly stable. The nonlinear stability of magnetic Bénard problem has been studied with different methods, and is based on the choice of an appropriate Lyapunov functional. The original energy method furnishes for $\mathcal{R}_n$ the same critical number as for a non electrically conducting fluid. We shall call "classical energy" the Lyapunov functional constituted only by the L^2 norms of the variables. The first heuristic reasoning that gives a $\mathcal{R}_n$ as an increasing function of Q_n is due to Galdi 1985 [2]. Then, a more general method of construction has been proposed by Galdi and Padula 1990 [3], and finally Rionero and Mulone [5] have considered a Lyapunov function that improves the earlier nonlinear stability results. Such methods, also called "generalized energy methods", require the use of a heavy norm in the energy functional containing the derivatives of the basic variables, also called "generalized energy". Till the paper [2], the only results which

let the temperature gradient to increase with the increasing of the basic magnetic field where those due to Chandrasekhar for the linearized system.

The aim of this note is to furnish a naive Lyapunov functional that allows the Rayleigh number to be an increasing function of the Hartman number and that employs only weak L^2 norms of the variables. Actually, we observe that, in order to qualitatively show the inhibiting effect of the magnetic field on the convection, it is sufficient to add in the energy functional an additional (multiplicative) term in the variables which can be dominated by the classical energy functional. Our scope is not to try improving the earlier results, we only wish to propose a new method in the study of nonlinear stability that we find useful and somewhat simpler than those described above.

2 New naive Lyapunov functional

Let us consider a horizontal layer of electrically conducting viscous fluid heated from below upon which an uniform magnetic field H orthogonal to the layer is impressed. Let us denote by s_0 the state in which a steady adverse temperature gradient is mantained and there is no motion, and by $\boldsymbol{u}$, p, θ and $\boldsymbol{h}$ the perturbations to the velocity, pressure, temperature and induced magnetic field of the state s_0.

According to [1], the perturbation $(\boldsymbol{u}, p, \theta, \boldsymbol{h})$ obeys to the following adimensionalized equations:

$$
\begin{gathered}
\nabla \cdot \boldsymbol{u} = 0 \\
\frac{\partial \boldsymbol{u}}{\partial t} + R\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \Delta \boldsymbol{u} - \nabla p + R\theta \boldsymbol{k} + Q\boldsymbol{k} \cdot \nabla \boldsymbol{h} + +Q\boldsymbol{h} \cdot \nabla \boldsymbol{h} \\
Pr\frac{\partial \theta}{\partial t} + PrR\boldsymbol{u} \cdot \nabla \theta = \Delta \theta + Rw \\
\nabla \cdot \boldsymbol{h} = 0 \\
Pm\frac{\partial \boldsymbol{h}}{\partial t} + PmR\boldsymbol{u} \cdot \nabla \boldsymbol{h} = \Delta \boldsymbol{h} + Q\boldsymbol{k} \cdot \nabla \boldsymbol{u} + Q\boldsymbol{h} \cdot \nabla \boldsymbol{u}
\end{gathered}
\tag{2.1}
$$

where $R^2 = \alpha\beta g d^4/\chi\nu$ is the *Rayleigh number*, α the linear thermal dilatation coefficient, β the value of the temperature gradient of the steady state s_0, g the acceleration of gravity, ν the kinematic viscosity, $Pr = \nu/\kappa$ the *Prandtl number*, κ the coefficient of thermal diffusivity, $Q^2 = \mu_m H^2 d^2/4\pi\varrho\nu\eta_m$ the *Chandrasekhar number*, μ_m the magnetic permeability, η_m the magnetic resistivity, ϱ the (constant) density and $P_m = \nu/\eta_m$ the *magnetic Prandtl number*. Finally, by p we denote both the mechanical and the magnetic pressures. To (2.1) we add the stress-free boundary conditions for $\boldsymbol{u}$ and θ:

$$
\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = 0 \tag{2.2}
$$

at $z = 0, 1$, and moreover:

$$
\boldsymbol{h} \times \boldsymbol{k} = \boldsymbol{0} \tag{2.3}
$$

at $z = 0, 1$. To exclude rigid motions and constant vertical magnetic fields we impose the conditions:

$$\int_C u = \int_C v = \int_C \boldsymbol{h} \cdot \boldsymbol{k} = 0 \tag{2.4}$$

where C is a periodicity cell.

It is well known from the experiments that the presence of a magnetic field has a stabilizing effect.

Let us apply the energy method in the classical way. Let us consider the scalar product in $L^2(C)$ of $(2.1)_2$ by $\boldsymbol{u}$, of $(2.1)_3$ by θ, of $(2.1)_5$ by $\boldsymbol{h}$. Adding the three new equations we have:

$$\frac{1}{2}\frac{d}{dt}E + D = 2R(\theta, w) \tag{2.5}$$

with

$$\begin{aligned} E(t) &= \|\boldsymbol{u}\|^2 + Pr\|\theta\|^2 + Pm\|\boldsymbol{h}\|^2 \\ D(t) &= \|\nabla \boldsymbol{u}\|^2 + \|\nabla\theta\|^2 + \|\nabla \boldsymbol{h}\|^2. \end{aligned} \tag{2.6}$$

It is clear that it is necessary to modify the energy functional in order to take into account the magnetic effect.

According to [3], for small initial data we can disregard the nonlinear terms in the construction of the Lyapunov functional. Therefore, we rewrite only the linear part of (1) as:

$$\begin{aligned} &\nabla \cdot \boldsymbol{u} = 0 \\ &\frac{\partial \boldsymbol{u}}{\partial t} = \Delta \boldsymbol{u} - \nabla p + R\theta \boldsymbol{k} + Q\boldsymbol{k} \cdot \nabla \boldsymbol{h} \\ &Pr\frac{\partial \theta}{\partial t} = \Delta\theta + Rw \\ &\nabla \cdot \boldsymbol{h} = 0 \\ &Pm\frac{\partial \boldsymbol{h}}{\partial t} = \Delta \boldsymbol{h} + Q\boldsymbol{k} \cdot \nabla \boldsymbol{u} \end{aligned} \tag{2.7}$$

Let us consider the third component of the vector equation $(2.7)_5$ and let us integrate it from 0 to z. Multiplying scalarly in $L^2(C)$ the resulting equation by θ, using $(2.7)_4$ and the boundary conditions, we obtain:

$$Pm\left(\theta, \frac{\partial}{\partial t}\int_0^z h_3\right) = \left(\theta, \Delta\int_0^z h_3\right) + (\theta, Qw)\ . \tag{2.8}$$

Let us now consider the scalar product in $L^2(C)$ of $(2.7)_3$ by $\int_0^z h_3$. Multiplying both sides by Pm/Pr, using again (2.2) and (2.3), we have:

$$Pm\left(\frac{\partial\theta}{\partial t}, \int_0^z h_3\right) = \frac{Pm}{Pr}\left(\Delta\theta, \int_0^z h_3\right) + R\frac{Pm}{Pr}\left(w, \int_0^z h_3\right)\ . \tag{2.9}$$

Adding (2.8) to (2.9) we obtain:

$$Pm\frac{d}{dt}\left(\theta, \int_0^z h_3\right) = -\left(1+\frac{Pm}{Pr}\right)\left(\nabla\theta, \nabla\int_0^z h_3\right) + Q(\theta, w) + R\frac{Pm}{Pr}\left(w, \int_0^z h_3\right) \tag{2.10}$$

The new energy functional we propose is obtained by subtracting from (2.5) the previous equation multiplied by a factor γ:

$$\frac{1}{2}\frac{d}{dt}\mathcal{E} + \mathcal{D} = (2R - \gamma Q)(\theta, w) - \gamma R\frac{Pm}{Pr}\left(w, \int_0^z h_3\right) \tag{2.11}$$

with

$$\begin{aligned} \mathcal{E} &= E - 2\gamma Pm\left(\theta, \textstyle\int_0^z h_3\right) \\ \mathcal{D} &= D - \gamma(1+b)\left(\nabla\theta, \nabla \textstyle\int_a^z h_3\right), \end{aligned} \tag{2.12}$$

with $b = \dfrac{Pm}{Pr}$. Using Poincaré inequality, see [4],

$$\begin{aligned} \pi\|\boldsymbol{u}\|^2 &\le \|\boldsymbol{u}\|^2 \\ \pi\|\nabla\boldsymbol{u}\|^2 &\le \|\Delta\boldsymbol{u}\|^2 \end{aligned} \tag{2.13}$$

we obtain:

$$\begin{aligned} \mathcal{E} &\ge \|\boldsymbol{u}\|^2 + Pr\|\theta\|^2 + Pm\|\boldsymbol{h}\|^2 - 2\frac{\gamma Pm}{\pi}\|\theta\|\|\boldsymbol{h}\| = \mathcal{E}_I \\ \mathcal{D} &\ge \pi^2\|\boldsymbol{u}\|^2 + \pi^2\left(1 - \frac{\gamma(1+b)}{2\pi}\right)\|\theta\|^2 + \pi^2\left(1 - \frac{\gamma(1+b)}{2\pi}\right)\|\boldsymbol{h}\|^2 = \mathcal{D}_1 . \end{aligned} \tag{2.14}$$

The factor γ has to satisfy the condition:

$$\gamma^2 < \min\left\{\frac{\pi^2}{b}, \frac{\pi^2}{(1+b)^2}\right\} = \frac{\pi^2}{(1+b)^2}, \tag{2.15}$$

in order to render $\mathcal{E}_I$ and $\mathcal{D}_1$ positive definite.

Then we have

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_I + \mathcal{I} \le 0, \tag{2.16}$$

with

$$\mathcal{I} = \pi^2\|\boldsymbol{u}\|^2 + \pi^2\left(1 - \frac{\gamma(1+b)}{2\pi}\right)\|\theta\|^2 + \pi^2\left(1 - \frac{\gamma(1+b)}{2\pi}\right)\|\boldsymbol{h}\|^2 - |2R - \gamma Q|\|\theta\|\|\boldsymbol{u}\| - \frac{\gamma Rb}{\pi}\|\boldsymbol{u}\|\|\boldsymbol{h}\|. \tag{2.17}$$

3 Sufficient condition for nonlinear stability

In order to obtain stability, $\mathcal{I}$ has to be positive definite. It's sufficient that the determinant of the matrix associated to $\mathcal{I}$, say $A_{\mathcal{I}}$ is positive, since it is easy to see that all the submatrices have positive determinant. Then, putting $detA_{\mathcal{I}} > 0$, we obtain a second order inequality in γ:

$$(R^2b^2 + \pi^2 Q^2)\gamma^2 - 2[2\pi^2 RQ - \pi^5(1+b)]\gamma - 4\pi^2(\pi^4 - R^2) < 0 . \tag{3.1}$$

This equation has real roots only if its discriminant is non-negative, say $\Delta \geq 0$. Then, a necessary condition for the existence of at least one γ satisfying (3.1) is:

$$4b^2R^4 - 4\pi^4 b^2 R^2 + 4(1+b)(\pi^2 R)(\pi^3 Q) - \pi^6[4Q^2 - \pi^2(1+b)^2] < 0 . \tag{3.2}$$

A sufficient condition that ensures that (3.2) is satisfied is

$$4b^2R^4 - 2\pi^4[b^2 - 1 - 2b]R^2 - \pi^6[2Q^2 + \pi^2(1+b)^2] < 0 , \tag{3.3}$$

that implies:

$$R^2 < \left[\frac{\pi^4[b^2-1-2b]}{4b^2} + \sqrt{\frac{\pi^8[5b^4+4b^3+6b^2+1+4b]}{16b^4} + \frac{\pi^6 Q^2}{2b^2}}\right] . \tag{3.4}$$

Notice that, denoting with $\gamma_{\pm}$ the two roots of (3.3), the condition $\Delta \geq 0$ ensures also that $\gamma_- < \frac{2\pi}{(1+b)}$. We observe that the first rooth γ_- of (3.3) is always negative, then condition $\Delta > 0$ ensures that whenever R satisfies (), there exists a γ less than the minimum between $\pi/(1+b)$ and γ_+, such that the functional $\mathcal{I}$ is positive definite. Therefore, applying the reasoning of [3], we conclude that there is an exponential nonlinear decay to the rest state for small initial data. It is not difficult to see that, at least asymptotically, this relation is satisfied for Rayleigh numbers increasing as the square root of the Hartman number Q. These relations, even though obtained in a very naive way, should be compared with those previously obtained in [2], [3], [5]. If $b < 1$, better conditions can be deduced as a coincidence between linear $\mathcal{R}_L$ and $\mathcal{R}_n$ for bounded Q, however these results are beyond the scopes of this note.

We still emphasize that our study is far to be complete for the following reasons:

1 we have not computed the maximum of the functional $\mathcal{I}$ in the class of the functions $\mathbf{u}$, $\mathbf{h}$, θ, where the first two vector fields are solenoidal;

2 instead of solving the correct condition $\Delta > 0$, we have just found some very elementary sufficient conditions for nonlinear stability.

For these reasons we don't make any comparison between the graphs obtained in earlier works. We limit ourselves to remark that $\mathcal{R}_n$ behaves asymptotically like the square root of Q, in the line with all nonlinear results already proven. This result let us to hope that a more precise and detailed study could furnish results analogous to those of [5] but in a shorter way.

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Stability of Navier-Stokes flows through permeable boundaries

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1 Introduction

A phenomenon observed in non-Newtonian fluids which is absent in Navier-Stokes fluids, is the presence of a normal stress component, additional to the pressure, determined by the velocity field. For second grade incompressible fluids which adhere to a boundary this component has been calculated explicitly by Berker [1], and the expression shows that for an incompressible Navier-Stokes flow, the component is zero.

A question to be asked then is: if the boundary is permeable *i.e.* allows the motion of fluid through it, is it possible for fluid to pass through the boundary as the result of fluid motion in the region bounded by the particular wall? A model for studying this situation has, for incompressible fluids of second grade, been proposed and studied in [6, 7]. It was found that for second grade fluids the problem is well-posed if certain additional boundary conditions were imposed. The additional boundary conditions can be expected since the equations of motion of second grade fluids are third order.

The model mentioned above, is formulated for arbitrary fluids and the question arises as to its applicability for Navier-Stokes fluids, which are less complex than the nonlinear fluids. Once this is established, the behaviour at the permeable boundary can be compared to what, seemingly, has been observed experimentally. It is the aim of this paper to establish at least the aspect of well-posedness which deals with stability of the rest state. The more delicate question of existence will be dealt with in subsequent work.

2 The model

In this section we shall give a somewhat detailed overview of the model for permeable flow, since it has bearing on the meaning of the results. For full details [7] should be consulted.

Consider a container filled with fluid into which another fluid-filled container with a permeable wall is immersed. The 3-dimensional space between the two containers is denoted by Ω and the interior of the immersed container is denoted by Ω_0. The permeable interface boundary is denoted by Γ and the outer boundary of Ω is denoted by Σ. This boundary is supposed to be impermeable and sticky. A visual representation of the 'geometry' is given in Figure 1 below.

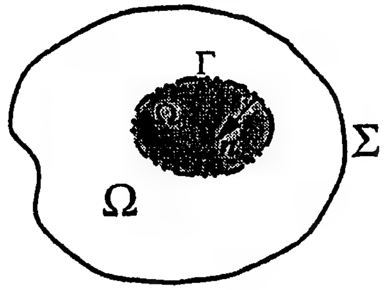

Figure 1: Geometry of the model

The unit exterior normal to Γ will be denoted by $\boldsymbol{n}$, and the trace operator γ_0 will be used to denote restriction to Γ. Let $\boldsymbol{v}(x,t)$ and $p(x,t)$ denote the velocity and pressure fields at $x \in \Omega$ at time $t > 0$. The equations of motion we shall be interested in are

$$\left.\begin{array}{l} \rho D_t \boldsymbol{v}(x,t) = \nabla \cdot \boldsymbol{T}(p,\boldsymbol{v}) \\ \nabla \cdot \boldsymbol{v}(x,t) = 0, \end{array}\right\} \tag{1}$$

where $\rho > 0$ denotes the constant density of the fluid, and $\boldsymbol{T}(p,\boldsymbol{v})$ the stress tensor. The *material derivative* D_t is defined as

$$D_t := \partial_t + \boldsymbol{v} \cdot \nabla. \tag{2}$$

In the case of a Navier-Stokes fluid,

$$\boldsymbol{T}(p,\boldsymbol{v}) := -p\boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{v}) \tag{3}$$

with $\mu > 0$ the viscosity, which is assumed to be constant, and $\boldsymbol{D}(\boldsymbol{v})$ the *rate of deformation tensor*, *i.e.*

$$\boldsymbol{D}(\boldsymbol{v}) := \tfrac{1}{2}[\nabla \boldsymbol{v} + \nabla^T \boldsymbol{v}]. \tag{4}$$

Here the notation $\nabla^T \boldsymbol{v}$ is used to denote the transpose (adjoint) of the tensor $\nabla \boldsymbol{v}$. It is assumed that the velocity field $\boldsymbol{v}$ always satisfies the homogeneous Dirichlet boundary condition

$$\boldsymbol{v}(\cdot,t) = 0 \text{ on } \Sigma \text{ for } t > 0.$$

The interesting part of the modelling concerns the boundary condition at Γ. It is assumed that the flow through the permeable boundary is always in the direction of the normal, *i.e.*

$$\gamma_0 v = -\eta n \text{ on } \Gamma. \tag{5}$$

The scalar-valued function η defined on Γ is unknown, and is determined by a *dynamic boundary condition* which, like the first equation in (1), is an evolution equation obtained by formulating the dynamics of the flow through the permeable boundary. The permeability of the boundary Γ is brought into play by means of the concept of *effective area*. Heuristically, if the permeable boundary is thought of as consisting of 'holes' through which fluid can 'permeate', the area through which fluid can move, is smaller than the 'true' area of the surface in question. This should be seen as a local property. Thus the measure of effective area da is related to the measure ds of true area by

$$da = \zeta(y)\,ds;\ y \in \Gamma \tag{6}$$

with ζ a measurable function defined on Γ such that $0 < \zeta(y) < 1$ for every $y \in \Gamma$. The *momentum transfer tensor* is defined as

$$P := \rho\gamma_0 v \otimes \gamma_0 v. \tag{7}$$

We use the 'o-times' notation for tensor products.

The other concept of importance is that of *surface density* of the fluid in the surface Γ. Heuristically this is given by the expression $\sigma(y) = \delta(y)\rho\,\zeta(y)$ with $\delta(y)$ a measure of the 'thickness' of the surface at y. Formally, it expresses a way of calculating the mass of fluid in any measurable boundary patch as an integral with respect to the measure ds. We shall assume that *σ is a measurable function which is bounded and bounded away from zero by a positive number.*

The law of conservation of linear momentum in the surface Γ is, in the absence of body forces, expressed in terms of the momentum transfer tensor P and the stress tensor T. It is stated in the following way: *for every measurable boundary patch $\Gamma' \subset \Gamma$, the rate of change of linear momentum in Γ' is explained by the net influx of momentum into Γ' and the resultant force due to stress on the patch.* Thus, if P_0 and T_0 represent the momentum flux tensor and the stress tensor on the Ω_0-side, then

$$\frac{d}{dt}\int_{\Gamma'} \sigma(y)\gamma_0 v(y,t)\,ds(y) = \int_{\Gamma'} [P - P_0]n\,da + \int_{\Gamma'} [T_0 - T]n\,ds,$$

because momentum can only be transported through 'holes'. Since this holds for arbitrary patches Γ', it formally follows that

$$\partial_t[\sigma\gamma_0 v] - \zeta[P - P_0]n = [T_0 - T]n \tag{8}$$

where we have used (6) to obtain the factor ζ. Substitution of (5) in (8), yields

$$[\sigma\partial_t\eta]n + \zeta P n - T n = -T_0 n + \zeta P_0 n. \tag{9}$$

Next, we assume that *the fluid in Ω_0 is at rest.* This implies that in Ω_0, the pressure p_0 depends only on time, and from (7), $\boldsymbol{P}_0$ is the zero tensor. It follows from (9) that

$$\sigma\partial_t\eta + \zeta\boldsymbol{n}\cdot\boldsymbol{P}\boldsymbol{n} - \boldsymbol{n}\cdot\boldsymbol{T}\boldsymbol{n} = -p_0(t). \tag{10}$$

This is the general form of the *dynamic boundary condition.* It is independent of the choice of $\boldsymbol{T}$ and therefore applies to any fluid.

The final step is to express the rate of deformation tensor at the boundary Γ in terms of the unknown function η. Differential geometric considerations [6, 7] show that this is possible. It may be shown that

$$\gamma_0[\boldsymbol{D}(\boldsymbol{v})] = -\eta\boldsymbol{M} + \tfrac{1}{2}\boldsymbol{N} \tag{11}$$

where the tensors $\boldsymbol{M}$ and $\boldsymbol{N}$ are defined by

$$\boldsymbol{M} := \kappa\boldsymbol{n}\otimes\boldsymbol{n} - \kappa_1\tau_1\otimes\tau_1 - \kappa_2\tau_2\otimes\tau_2 \tag{12}$$
$$\boldsymbol{N} := \boldsymbol{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\boldsymbol{n}. \tag{13}$$

In (12) τ_1 and τ_2 are unit vectors tangential to orthogonal normal curves in Γ, κ_1 and κ_2 the corresponding curvatures, and $\kappa := \kappa_1 + \kappa_2$ the mean curvature. It will be assumed throughout that $\Omega\cup\Omega_0$ *is a bounded set and that the surface* Γ *is infinitely smooth.* This implies amongst other things that the mean curvature is a very smooth function. In (13) the vector $\boldsymbol{\psi}$ is tangential to Γ. It can indeed be expressed in the form

$$\boldsymbol{\psi} = \boldsymbol{\omega}\wedge\boldsymbol{n} - 2\nabla_s\eta, \tag{14}$$

with $\boldsymbol{\omega} := \nabla\wedge\boldsymbol{v}$ the vorticity, and ∇_s the *surface gradient.* The wedge notation is used for vector products. A direct application of Stokes' theorem shows that $\boldsymbol{\omega}$ is tangential because of (5). It follows from (5) and (7), that $\boldsymbol{n}\cdot\boldsymbol{P}\boldsymbol{n} = \rho\eta^2$. In addition, it is seen from (11), (12) and (13), that $\gamma_0[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n} = -2\kappa\eta\boldsymbol{n} + \boldsymbol{\psi}$, so that

$$\boldsymbol{n}\cdot\gamma_0[\boldsymbol{D}(\boldsymbol{v})]\boldsymbol{n} = -\kappa\eta.$$

By making use of this result and (3), it is seen that $\boldsymbol{n}\cdot\boldsymbol{T}\boldsymbol{n} = -\gamma_0 p - 2\mu\kappa\eta$ so that the (10) takes the form

$$\sigma\partial_t\eta + \zeta\rho\eta^2 + 2\mu\kappa\eta + \gamma_0 p = -p_0(t). \tag{15}$$

We shall refer to this as the *dynamic boundary condition.*

3 The system as an implicit equation

The system consisting of (1) and (15) may very easily be viewed as a very mildly coupled system with the coupling situated in the pressure term which appears in both equations. Apart from that, there is an amount of coupling within the system

(1) due to the constraint that $\nabla \cdot v = 0$. The system is, indeed more closely coupled once one observes that

$$\eta = \eta_v := -[\gamma_0 v] \cdot n .$$

The modelling described in the previous section therefore leads to the following constraints on the velocity field v:

$$\left.\begin{aligned} &\nabla \cdot v = 0, \\ &v = o \text{ on } \Sigma, \\ &\gamma_0 v = -\eta_v n \text{ on } \Gamma, \\ &n \cdot [\gamma_0 D(v)] n = -\kappa \eta_v . \end{aligned}\right\} \tag{16}$$

It is important to note that these constraints are all of a linear nature and that a large class of functions has all these properties, *e.g.* the class of all solenoidal $C_0^\infty(\Omega)$-functions.

Let us write down explicitly the system of evolution equations alluded to above, in a form slightly different to that obtained from (1), (2), (3), (4) and (15):

$$\left.\begin{aligned} &\rho^{1/2}[\partial_t v + v \cdot \nabla v] - 2\rho^{-1/2}\mu \nabla \cdot D(v) + \rho^{-1/2}\nabla p = o \\ &\sigma^{1/2}\partial_t \eta_v + \zeta \sigma^{-1/2}\rho \eta_v^2 + 2\mu\sigma^{-1/2}\kappa\eta_v + \sigma^{-1/2}\gamma_0 p = \sigma^{-1/2}p_0(t). \end{aligned}\right\} \tag{17}$$

This system may be expressed as an *implicit evolution equation* by defining the mappings L, B, N and L formally by

$$\left.\begin{aligned} Lv &:= -2\mu\langle \rho^{-1/2}\nabla \cdot D(v), -\sigma^{-1/2}\kappa\eta_v \rangle \\ Bv &:= \langle \rho^{1/2} v, \sigma^{1/2}\eta_v \rangle \\ N(v) &:= \langle \rho^{1/2} v \cdot \nabla v, \zeta\rho\sigma^{-1/2}\eta_v^2 \rangle \\ \mathrm{L}\, p &:= \langle \rho^{-1/2}\nabla p, \sigma^{-1/2}\gamma_0 p \rangle . \end{aligned}\right\} \tag{18}$$

The notation $\langle a, b\rangle$ is used to denote an ordered pair, *i.e.* an element of some product space. The mappings L, B and L have the potential of becoming linear once the class of functions on which they act, is prescribed as some vector space. This can never happen to N. The system (17) can now be rewritten in the implicit form

$$\partial_t[Bv] + Lv + \mathrm{L}\, p = \langle o, p_0(t) \rangle = \mathrm{L}\, p_0(t). \tag{19}$$

In this equation the velocity field v and the pressure p are the only unknown functions.

In order to be more explicit about the mappings defined in (18) and the equation (19), we define the following spaces:

- X denotes the space $L^2(\Omega)$ of measurable square integrable vector fields defined on Ω. We shall not be too concerned about the topology induced by the 'natural' norm $\| \ \|_X$ which makes X a Banach space.

- $\mathfrak{D}$ will denote the vector subspace of X consisting of those elements of the Sobolev space $H^2(\Omega)$ which satisfy the constraints (16). According to the trace theorem [5], the constraints make sense in $H^2(\Omega)$.
- $Y := L^2(\Omega) \times L^2(\Gamma)$ endowed with the inner product

$$(\langle a,b\rangle, \langle a',b'\rangle)_Y := (a,a')_X + (b,b')_{L^2(\Gamma)}.$$

The corresponding norm will be denoted by $\|\ \|_Y$. With these endowments, Y becomes a Hilbert space. By the boundedness-assumptions on the surface density σ, the metric in $L^2(\Gamma)$ can, without any complications be defined with σ as a weight. We shall have occasion to use the equivalent norm

$$(f,g)_\sigma := \int_\Gamma \sigma(y)f(y)g(y)\,ds(y)$$

for $L^2(\Gamma)$.

The space $\mathfrak{D}$ shall be taken as the domain for the operators defined in (18). By the trace theorem, L and B are well-defined linear operators on $\mathfrak{D}$ with range in Y. By the Sobolev embedding theorem, the nonlinear operator N is also defined on $\mathfrak{D}$ with range in Y.

We shall use the notation $v(t)$ for the function $v(t) : x \in \Omega \mapsto v(x,t)$ for $t > 0$. Similar notation will be used for the pressure p. Let $v : t > 0 \mapsto v(t) \in \mathfrak{D}$ be a given function. We shall say that Bv is *differentiable in* Y if for every $t > 0$, $\lim_{h\to 0} h^{-1}[Bv(t+h) - Bv(t)]$ exists in the norm of Y. This limit will be denoted by $[Bv(t)]'$. The pair of functions $v(t) \in \mathfrak{D}$, $p(t) \in H^1(\Omega)$ is said to be a *solution* of the *Implicit Cauchy Problem (ICP)* if $Bv(t)$ is differentiable in Y and for given $p_0(t)$ and $y \in Y$,

$$\left.\begin{array}{l} [Bv(t)]' + Lv(t) + N(v(t)) + \mathrm{L}p(t) = \mathrm{L}\,p_0(t) \\ \lim\limits_{t\to 0+} \|Bv(t) - y\|_Y = 0. \end{array}\right\} \tag{20}$$

4 A Helmholtz-Weyl projection

The traditional Helmholtz decomposition of vector fields is associated with an orthogonal projection which annihilates pressure gradients and leaves certain solenoi dal vectors intact. Since the implicit system (20) contains terms of the form L p, this is of little or no use in the present situation. There exists, however, an orthogonal projection operator $P : Y \to Y$ which annihilates terms of the form L p and, for vector fields v which comply with the constraints (16), leaves Bv intact [8]. This projection, applied to (20), gives the *Projected implicit Cauchy Problem (PICP)*

$$\left.\begin{array}{l} [Bv(t)]' + PLv(t) + PN(v(t)) = 0 \\ \lim\limits_{t\to 0+} \|Bv(t) - Py\|_Y = 0. \end{array}\right\} \tag{21}$$

This formulation concerns only the velocity field v. The mapping $PL : \mathfrak{D} \subset X \to Y$ is sometimes called a *Stokes-like* operator.

5 Some identities and inequalities

Before proceeding with the study of stability, a few identities derived from integrati on by parts, and some inequalities need to be discussed. The following identities are obtained after integration by parts, the definitions (18) and the fact that $PB\varphi = B\varphi$ if $B\varphi \in \mathfrak{D}$:

PROPOSITION 5.1. *For* φ, $\psi \in \mathfrak{D}$,

$$(PL\varphi, B\psi)_Y = 2\mu(D(\varphi), D(\psi)) \tag{22}$$

$$(PN(\varphi), B\varphi)_Y = \rho \int_\Gamma [\zeta - \tfrac{1}{2}]\eta_\varphi^3 \, ds \tag{23}$$

where

$$(D(\varphi), D(\psi)) := \int_\Omega D(\varphi) : D(\psi) \, dx.$$

The colon-notation is used for the 'scalar product' between tensors, *i.e.* the sum of products of corresponding components.

By expanding the expression for $D(\varphi) : D(\varphi)$ in terms of the definition (4), performing integration by parts on the 'odd' term, and making use of (11), the following is obtained:

PROPOSITION 5.2. *For* $\varphi \in \mathfrak{D}$,

$$||D(\varphi)||^2 = \tfrac{1}{2}\left[||\nabla\varphi||^2 + \int_\Gamma \kappa(y)\eta_\varphi^2(y)\, ds(y)\right]. \tag{24}$$

This identity has important consequences when the mean curvature of the surface Γ is nonnegative. We observe first that since elements of $\mathfrak{D}$ vanish on the outer surface Σ and Ω is bounded, the Poincaré inequality holds:

PROPOSITION 5.3. (The Poincaré inequality.) *There exists a constant* $C_p > 0$ *such that for any* $\varphi \in \mathfrak{D}$,

$$||\nabla\varphi||^2 \geq C_p||\varphi||^2_{L^2(\Omega)}. \tag{25}$$

In order to keep the norm in the Sobolev space $H^1(\Omega)$ dimensionally correct, we shall define it by means of

$$||\varphi||^2_{H^1(\Omega)} := ||\nabla\varphi||^2 + C_p||\varphi||^2_{L^2(\Omega)}.$$

We may now prove the following:

LEMMA 5.4. *If the mean curvature* κ *of the surface* Γ *is nonnegative, then* $||D(\varphi)||$ *is a norm on* $\mathfrak{D}$, *equivalent to the norm of the space* $H^1(\Omega)$.

Proof. From (24) and (25) we see that

$$||D(\varphi)||^2 \geq \tfrac{1}{2}||\nabla\varphi||^2 \geq \tfrac{1}{4}[||\nabla\varphi||^2 + C_p||\varphi||^2_{L^2(\Omega)}] = \tfrac{1}{4}||\varphi||^2_{H^1(\Omega)}. \tag{26}$$

The other inequality follows from (24) by making use of the trace theorem. □

We shall also need estimates of $PL\varphi$ in terms of the norm in the Sobolev space $H^2(\Omega)$. This is obtained from the standard *a-priori* estimate (which holds when the boundary is smooth)

$$\begin{aligned}\|\varphi\|^2_{H^2(\Omega)} &\leq C_\Delta\|\Delta\varphi\|^2_{L^2(\Omega)} + C_\gamma\|n\cdot\gamma_0\varphi\|^2_{L^2(\Gamma)}\\ &= C_\Delta\|\Delta\varphi\|^2_{L^2(\Omega)} + C_\gamma\|\eta_\varphi\|^2_{L^2(\Gamma)},\end{aligned} \tag{27}$$

for $\varphi \in \mathfrak{D}$ (see *e.g.* [2]). The right of (27) may, in the case where κ is nonnegative, be compared to the expression

$$\begin{aligned}\|L\varphi\|^2_Y &= \tfrac{\mu^2}{\rho}\|\Delta\varphi\|^2_{L^2(\Omega)} + \mu^2\|\tfrac{\kappa}{\sigma^{1/2}}\eta_\varphi\|^2_{L^2(\Gamma)}\\ &\geq \tfrac{\mu^2}{C_\Delta\rho}[C_\Delta\|\Delta\varphi\|^2_{L^2(\Omega)}] + \tfrac{k^2\mu^2}{C_\gamma S}[C_\gamma\|\eta_\varphi\|^2_{L^2(\Gamma)}]\end{aligned} \tag{28}$$

with $k^2 := \inf\kappa^2(y)$ and $S := \sup\sigma(y)$. Combination of (27) and (28) yields the estimate

$$\|L\varphi\|^2_Y \geq \mu^2\min\{\tfrac{1}{C_\Delta\rho}, \tfrac{k^2}{C_\gamma S}\}\|\varphi\|^2_{H^2(\Omega)} \tag{29}$$

for $\varphi \in \mathfrak{D}$. We therefore have

LEMMA 5.5. *For $\varphi \in \mathfrak{D}$, the norm of the Sobolev space $H^2(\Omega)$ and $\|L\varphi\|_Y$ are equivalent.*

Proof. The converse of (29) follows from the trace theorem. □

We end this section with an estimate of the nonlinear operator N, which, as already remarked, maps $\mathfrak{D}$ into Y. For this we shall need the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, *i.e. there exists a positive constant C_E such that*

$$\sup_{y\in\Omega}|\varphi(y)| \leq C_E\|\varphi\|_{H^2(\Omega)} \tag{30}$$

for every $\varphi \in H^2(\Omega)$. We also need the embedding $\gamma_0[H^1(\Omega)] \hookrightarrow L^4(\Gamma)$ in the form: *there exists a positive constant $C_{E,4}$ such that*

$$\|\eta_\varphi\|_{L^4(\Gamma)} \leq C_{E,4}\|\varphi\|_{H^1(\Omega)} \tag{31}$$

which is true for every $\varphi \in \mathfrak{D}$. Finally, we define the constant C_L according to (27) and (29), by means of

$$\left.\begin{aligned}C_L &:= \min\{\tfrac{1}{C_\Delta\rho}, \tfrac{k^2}{C_\gamma S}\}\\ S &:= \sup_{y\in\Gamma}\sigma(y)\\ k^2 &:= \inf_{y\in\Gamma}\kappa^2(y).\end{aligned}\right\} \tag{32}$$

LEMMA 5.6. *Suppose that the mean curvature κ is nonnegative on Γ. Then there exists a constant $c_1 > 0$ such that for every $\varphi \in \mathfrak{D}$*

$$\|N(\varphi)\|_Y^2 \leq [c_1 \|D(\varphi)\| \, \|L\varphi\|_Y]^2$$

with c_1 defined by

$$\left.\begin{array}{l} c_1^2 := \frac{4\rho}{\mu^2 C_L^2}\left[C_E^2 + \frac{\rho C_{E,4}^4}{s}\right] \\ s := \inf_{y\in\Gamma} \sigma(y), \end{array}\right\}$$

with the embedding constants C_E and $C_{E,4}$ defined in (30) and (31).

Proof. From the definition of $N(\varphi)$,

$$\begin{aligned} \|N(\varphi)\|_Y^2 &= \rho \int_\Omega |\varphi \cdot \nabla\varphi|^2 \, dx + \rho^2 \int_\Gamma \frac{\zeta^2}{\sigma} \eta_\varphi^4 \, ds \\ &\leq \rho \int_\Omega |\varphi|^2 |\nabla\varphi|^2 \, dx + \frac{\rho^2}{s} \int_\Gamma \eta_\varphi^4 \, ds \\ &\leq \rho C_E^2 \|\varphi\|_{H^2(\Omega)}^2 \, \|\nabla\varphi\|^2 + \frac{\rho^2 C_{E,4}^4}{s} \|\varphi\|_{H^1(\Omega)}^4. \end{aligned}$$

The estimate is completed by using (26), (29) and (32), after writing the fourth power as the product of two squares. □

6 Stability

For the stability, we shall apply the results of [4] (which is an adaptation of the results of [Galdi-Padula, 3] to implicit evolution equations, making use of the theory of bilinear forms rather than spectral theory). To this end we define the bilinear forms R and S by means of

$$\begin{aligned} R(\varphi, \psi) &:= (PL\varphi, B\psi)_Y \\ S(\varphi, \psi) &:= (PB\varphi, B\psi)_Y. \end{aligned}$$

By Proposition 5.1, R is symmetric, and S is almost trivially so. Lemma 5.4 states that the quadratic form $\hat{R}$ associated with R is strictly positive. Following [4] we introduce the norms $\| \ \|_1$ and $\| \ \|_2$ on $\mathfrak{D}$ by

$$\begin{aligned} \|\varphi\|_1^2 &:= \hat{R}(\varphi) := R(\varphi, \varphi) \\ \|\varphi\|_2^2 &:= \hat{R}(\varphi) + \tfrac{1}{2}\|L\varphi\|_Y^2. \end{aligned}$$

By Lemma 5.4, $\| \ \|_1$ is equivalent to $\| \ \|_{H^1(\Omega)}$ if the mean curvature κ is nonnegative on Γ, and by Lemma 5.5, $\| \ \|_2$ is, under the same condition, equivalent to $\| \ \|_{H^2(\Omega)}$. Lemma 5.6 gives constants c_1 and $\alpha = 1$ such that for $\varphi \in \mathfrak{D}$,

$$\|N(\varphi)\|^2 \leq [c_1 \|\varphi\|_1^\alpha \|\varphi\|_2]^2$$

if $\kappa \geq 0$. The *energy* associated with $\varphi \in \mathfrak{D}$ is defined as

$$E_\varphi := \tfrac{1}{2}[\hat{S}(\varphi) + \hat{R}(\varphi)]$$

and is, by Lemma 5.4, seen to give rise to a norm equivalent to $\|\ \|_{H^1(\Omega)}$ provided, once again, that the mean curvature κ is nonnegative on γ. If $v(t)$ is a solution of the PICP (24) we shall use the notation

$$\begin{aligned} E(t) &:= E_{v(t)};\ t \geq 0 \\ E_0 &:= E(0). \end{aligned}$$

The stability theorem may now be obtained as a simple translation of one of the main theorems in [4].

THEOREM 6.1. (Stability.) *Suppose that the mean curvature κ is nonnegative on Γ. Let $v(t)$ is a solution of the Projected Implicit Cauchy Problem (24) which has the property that the mapping $t \mapsto PLv(t)$ is weakly continuous in Y. If*

$$E_0 < \tfrac{1}{2}\left[\tfrac{\sqrt{3}-1}{c_1}\right]^2,$$

then there exits a constant $C > 0$ such that for $t > 0$,

$$\|v(t)\|_1^2 \leq \tfrac{C}{1+t}.$$

7 Comparisons and remarks

In studies of stability it is always tempting to apply the 'well-tested' energy appro ach, *i.e.* to take the scalar product of (21) with Bv. The result, after making use of (22) and (23) is the energy identity

$$\tfrac{1}{2}\tfrac{d}{dt}[\rho\|v(t)\|^2_{L^2(\Omega)} + \|\eta_v(t)\|^2_\sigma] + 2\mu\|D(v)\|^2 + \int_\Gamma [\zeta - \tfrac{1}{2}]\eta_v^3\, ds = 0, \tag{33}$$

which is quite a fancy expression. There is one difficulty, however. The 'energy' is only in terms of L^2-norms of the velocity field and its normal component at the boundary. It becomes hard to estimate the deformation energy term from below in terms of this 'energy', but it can be done when it is assumed that the mean curvature κ is bounded below by a positive number [6]. In that case a Poincaré inequality of the form

$$\|D(v)\|^2 \geq c_p[\rho\|v\|^2_{L^2(\Omega)} + \|\eta_v\|^2_\sigma]$$

is possible. This is not the main difficulty, though. It lies in the boundary integral term which can only be estimated from above, and the only in terms of the norm of the Sobolev space $H^1(\Omega)$. Thus, except in the very special case when $\zeta(y) = 1/2$ on Γ, there is no hope of finding a stability result in this manner.

The energy method described above, works in the case of second grade fluids when it is assumed additionally that the tangential vector ψ defined in (14) is zero. In that case the energy (the term differentiated with respect to time) also contains the energy associated with the rate of deformation because the graded fluids are of *differential type*. The estimate of the boundary integral term, which is similar to the one in (33), then works in tune with the other terms to yield exponential decay of the energy [6].

A modified energy method in which the scalar product is taken with $Bv + Lv$, formally yields the same as the more sophisticated approach in [3, 4] except that differentiability of v in $H^1(\Omega)$ is required. This may intuitively be thought of as a substitute for the assumption of weak continuity in Theorem 6.1.

The nonnegativity of the mean curvature κ seems to be important in the results presented above and the others we have mentioned. The physical relevance of this assumption is unclear. It may have to do with the modelling assumption that the fluid in the container Ω_0 is at rest which means that fluid particles moving across the interface are accelerated/decelerated from/to rest over a very thin boundary. Studies of instability may throw some light on this mystery.

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On Steady Solutions of the Kuramoto-Sivashinsky equation

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Abstract

This article surveys our recent studies on the Kuramoto-Sivashinsky equation. Main topics include: (i) a simple proof of the nonexistence of monotonic global solutions; (ii) the existence of solutions which blow up on bounded intervals.

1 Introduction and results

This is a review on our recent researches for steady solutions of the Kuramoto-Sivashinsky equation and related problems [4][5].

The Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u_x^2 = 0, \qquad u = u(x,t) \tag{1}$$

is known to arize in a wide variety of interesting physical phenomena [8][9][11][13]. If we postulate $u(x,t) = -c^2 t + v(x)$, which is numerically suggested by [10] with the velocity $c \approx \sqrt{1.2}$, and define $y(x) = 2^{-1/2}c^{-1}v_x(2^{1/2}c^{-1}x)$, then we obtain the following third order ordinary differential equation (ODE)

$$\lambda y'''(x) + y'(x) = 1 - y(x)^2, \tag{2}$$

where $\lambda := c^2/2$. Solutions to (2) are customary referred to as steady solutions of the Kuramoto-Sivashinsky equation. We remark that (2) can be recovered from (1) by a usual traveling wave form $u(x,t) = v(x - ct)$ in (1) and suitable calculation. While if we just put $u(x,t) = v(x)$ in (1) and $y(x) = 2^{-1/2}c^{-1}v_x(2^{1/2}c^{-1}x)$, then we find that y verifies a similar ODE

$$\lambda y'''(x) + y'(x) = -y(x)^2, \tag{3}$$

where $\lambda = c^2/2$ as before.

Existence and/or nonexistence of various kinds of solutions to (2) have been investigated. Troy [15] made a thorough research of (2) in the case $\lambda = 1/2$; he imposes the initial condition $y(0) = 0$, $y'(0) = \beta > 0$, $y''(0) = 0$, and pursues the trajectory in detail regarding β as a parameter. Main results of [15] state that (2) with $\lambda = 1/2$ has at least two odd periodic solutions and that there further exist at least two globally bounded solutions obeying $y(x) \to \mp 1$ as $x \to \pm\infty$, respectively. In [16], the same method is applied to derive the nonexistence of monotonic global solutions; that is, there is no solution to (2) with $\lambda \approx 1/2$ such that $y'(x) > 0$ for all $x \in \boldsymbol{R}$ and $y(x) \to \pm 1$ as $x \to \pm\infty$, respectively. The nonexistence of monotonic global solutions is later extended in [6] to include all $\lambda > 0$. See also [12]. The method of proof in [6] should be compared with the one [1] given for a similar ODE

$$\varepsilon y'''(x) + y'(x) = \cos y(x), \qquad \varepsilon > 0, \tag{4}$$

which arises from dendritic crystal growth. After a pioneering observation by [7], it is now also concluded that there is no solution y of (4) which satisfies $y'(x) > 0$ for all $x \in \boldsymbol{R}$ and $y(x) \to \pm\pi/2$ as $x \to \pm\infty$, respectively. We remark that Toland [14] already gave an elementary approach to proving the nonexistence of monotonic solutions of (2) if $\lambda \geq 2/9$. Our first result [5], on the other hand, presents a simple unified argument toward the nonexistence of monotonic global solutions for certain class of ODEs including (2) and (4); however, [5] restricts the range of parameters λ and ε depending on nonlinearities. Here is our theorem.

Theorem 1. *If $\lambda \geq 8/27$ in (2) (resp. $\varepsilon \geq 32/27$ in (4)), then there is no solution y satisfying $y' > 0$ and $y(x) \to \pm 1$ (resp. $y(x) \to \pm\pi/2$) as $x \to \pm\infty$, respectively.*

We turn back to the existence of solutions: Jones, Troy, and MacGillivary [6] exhibit that there is an odd periodic solution to (2) for all small $\lambda > 0$. We again refer to [12] for the existence of a unique monotonic solution of (2) with $\lambda = 1/2$ on the half interval $0 \leq x < \infty$ which fulfills $y(0) = 0$ and $y(x) \to 1$ as $x \to \infty$.

Our second result [4] concerns another type of solutions to (2) from those mentioned above, which seems missing in the foregone literature. These are finite blow-up solutions in the sense that $y(x) \to \mp\infty$ as $x \to \pm x_\beta$, respectively, where positive x_β is finite. To be stated more precisely, we supplement (2) or (3) with the initial condition

$$y(0) = 0, \quad y'(0) = -\beta < 0, \quad y''(0) = 0. \tag{5}$$

Note that under (5) the solutions of (2) or (3) turn out to be odd. Now our results read as follows.

Theorem 2. *For all large values of $\beta > 0$, there exists finite $x_\beta > 0$ such that there are solutions y of (2)(5) veryfying $y'(x) < 0$ on $-x_\beta < x < x_\beta$ and*

$$y(x) \to \mp\infty \ \textit{as}\ x \to \pm x_\beta, \ \textit{respectively.}$$

Moreover there hold

$$\limsup_{x \uparrow x_\beta} (x_\beta - x)^3(-y(x)) \leq 30\sqrt{10}\lambda,$$
$$\liminf_{x \uparrow x_\beta} (x_\beta - x)^3(-y(x)) \geq 24\lambda.$$

Concerning (3)(5), there also exist similar blow-up solutions with possibly different x_β.

The proof of theorems involves the reduction of the third order equation into the second order one, which is assured by the monotonicity of the solution [3]. This technique, in this context, is introduced by Toland [14] and independently by Bernis, Peletier, and Williams [2].

2 Proof of theorems

First we prove Theorem 1 in the case (2); a similar obvious procedure applies well to (4). Since $y'(x) > 0$ for all $x \in \boldsymbol{R}$, we regard y as an independent variable. Let $x(y)$ denote the inverse function of $y(x)$ defined on $-1 \leq y \leq 1$. We put

$$t = y \qquad \text{and} \qquad v(t) = \{y'(x(t))\}^2 \tag{6}$$

and compute

$$v'(t) = 2y'(x(t))x'(t)y''(x(t)) = 2y''(x(t))$$
$$\lambda v''(t) = 2\lambda y'''(x(t))x'(t) = 2\frac{1-t^2}{\sqrt{v(t)}} - 2.$$

The problem we want to solve becomes the next overdetermined equation.

$$\begin{aligned} &\lambda v''(t) = 2\frac{1-t^2}{\sqrt{v(t)}} - 2, \quad v(t) > 0 \quad \text{for } -1 < t < 1, \\ &v(\pm 1) = v'(\pm 1) = 0. \end{aligned} \tag{7}$$

We prepare a comparison function

$$w(t) = M\sqrt{1-t^2},$$

where $M > 0$. Straightforward calculation tells us that $w(t)$ is a supersolution if $\lambda \geq 8/27$, irrespective to M. By virtue of $w(\pm 1) = 0$ and $w'(\pm 1) = \mp\infty$, we can assign M so that $w(t)$ touches $v(t)$ from above at the interior; a contradiction. This completes the proof of Theorem 1.

Next we construct the solutions claimed in Theorem 2 for the case (2). The same method works for (3). By symmetry it suffices to discuss the problem on the half interval $\{x < 0\}$. Since $y'(0) = -\beta < 0$, there is a small $x_0 > 0$ such that $y(x)$ is monotone decreasing on $-x_0 \leq x \leq 0$, where we can regard y as an independent

variable. Let $x(y)$ denote the inverse function of $y(x)$ defined on $0 \le y \le t_0 := y(x_0)$. We define $v(t)$ as in (6). Then the problem we want to argue becomes the following.

$$\begin{aligned} \lambda v''(t) &= -2\frac{1-t^2}{\sqrt{v(t)}} - 2, \\ v(0) &= \beta^2 > 0, \quad v'(0) = 0, \end{aligned} \tag{8}$$

where the differential equation is temporarily understood to hold on the interval $0 < t < t_0$. We also note that $y' < 0$ is taken into account. We proceed with series of lemmas.

Lemma 1. *For all sufficiently large $\beta > 0$, there exists a first $t_1 > 0$ such that*

$$v(t_1) > 0, \quad v'(t_1) = 0, \quad v''(t_1) > 0.$$

Proof. By virtue of $v'(0) = 0$ and $v''(0) = -2 + 1/\beta$, we infer that $v'(t) < 0$ at least for small $t > 0$ provided $\beta > 1/2$. So long as $v'(t) < 0$, we integrate (8) succesively, first on the interval $0 \le t \le 1$.

$$\begin{aligned} \lambda v'(t) &> -2\frac{t - t^3/3}{\sqrt{v(1)}} - 2t, \\ \lambda v(t) &> \lambda\beta^2 - \frac{t^2 - t^4/6}{\sqrt{v(1)}} - t^2, \end{aligned}$$

from which we especially see that $v(1) \ge \beta^2/4$. Now we integrate (8) on the interval $t \ge 1$, still keeping the assumption $v'(t) < 0$.

$$\begin{aligned} \lambda v'(t) &> \lambda v'(1) - \frac{4}{\beta}(t-1) + \frac{4}{3\beta}(t-1)^3 - 2(t-1), \\ \lambda v(t) &> \frac{\lambda}{4}\beta^2 + \lambda v'(1)(t-1) - \frac{2}{\beta}(t-1)^2 + \frac{1}{3\beta}(t-1)^4 - (t-1)^2. \end{aligned}$$

This shows that there must exist a first $t_1 > 0$ such that $v(t_1) > 0$ and $v'(t_1) = 0$, if we choose β large if necessary.

At t_1 there holds $v''(t_1) \ge 0$. If $v''(t_1) = 0$, then we discover that $v'''(t_1) > 0$ upon differentiating the equation (8), which implies that $v''(t) < 0$ in the left neighborhood of t_1; a contradiction with the definition of t_1. We have therefore $v''(t_1) > 0$ as desired. This completes the proof of Lemma 1. □

Lemma 2. *There holds that*

$$v(t) \ge v(t_1) \qquad \textit{for all } t \ge 0.$$

In particular the transformation (6) is legitimate for all $t \ge 0$.

Proof. Suppose that there exists a first $t_2 > t_1$ such that $v(t_2) = v(t_1)$. We multiply the equation (8) by $v'(t)$ and integrate on the interval $t_1 < t < t_2$.

$$\begin{aligned}\frac{\lambda}{2}(v'(t_2))^2 &= -4\int_{t_1}^{t_2}(1-t^2)(\sqrt{v(t)})'\,dt - 2(v(t_2)-v(t_1))\\ &= 4(t_2^2-t_1^2)\sqrt{v(t_1)} - 8\int_{t_1}^{t_2} t\sqrt{v(t)}\,dt\\ &< 4(t_2^2-t_1^2)\sqrt{v(t_1)} - 8\sqrt{v(t_1)}\int_{t_1}^{t_2} t\,dt = 0,\end{aligned}$$

where we have taken into account that $v(t) > v(t_1) = v(t_2)$ on $t_1 < t < t_2$. This is a contradiction and we arrive at $v(t) \geq v(t_1)$ for all $t \geq 0$. □

The behavior of $v(t)$ as $t \to \infty$ is described in the next lemma.

Lemma 3. *For all large t, there holds*

$$\frac{3}{10}3^{1/3}\lambda^{-2/3}t^{8/3} \leq v(t) \leq 3\cdot 3^{1/3}2^{-2/3}\lambda^{-2/3}t^{8/3}.$$

Proof. We introduce a functional

$$F[v(t)] = \frac{\lambda}{2}(v'(t))^2 + 2v(t) + 4(1-t^2)\sqrt{v(t)}.$$

We compute

$$\begin{aligned}\frac{d}{dt}F[v(t)] &= -8t\sqrt{v(t)} < 0 \qquad \text{for all } t > 0.\\ F[v(t_1)] &= 2v(t_1) + 4(1-t_1^2)\sqrt{v(t_1)}\\ &< 4v(t_1) + 4(1-t_1^2)\sqrt{v(t_1)} = -2\lambda v(t_1)v''(t_1) < 0.\end{aligned}$$

We thus obtain

$$\begin{gathered}\frac{\lambda}{2}(v'(t))^2 < 4t^2\sqrt{v(t)} - 4\sqrt{v(t)} - 2v(t) < 4t^2\sqrt{v(t)}\\ -\sqrt{\frac{8}{\lambda}}tv(t)^{1/4} < v'(t) < \sqrt{\frac{8}{\lambda}}tv(t)^{1/4} \qquad \text{for } t \geq t_1,\end{gathered}$$

from which we conclude that

$$\begin{aligned}v(t)^{3/4} &< \sqrt{\frac{9}{8\lambda}}(t^2-t_1^2) + v(t_1)^{3/4}\\ v(t) &< C_2t^{8/3},\end{aligned}$$

for all large t with a suitable constant C_2.

Now we observe that for all large t

$$\lambda v''(t) = -2\frac{1-t^2}{\sqrt{v(t)}} - 2$$
$$> \frac{2}{\sqrt{C_2}}t^{2/3} - \frac{2}{\sqrt{C_2}}t^{-4/3} - 2.$$

Integrating the above inequality twice, we find that

$$v(t) > C_1 t^{8/3},$$

for all large t with another appropriate constant C_1. With a little much careful analysis, we can determine the value of C_1 and C_2. This finishes the proof of Lemma 3. □

Lemma 3 immediately leads us to

$$x_\beta := \int_0^\infty \frac{dt}{\sqrt{v(t)}} < \int_0^{t_*} \frac{dt}{\sqrt{v(t_1)}} + \int_{t_*}^\infty \frac{dt}{\sqrt{C_1 t^{8/3}}} < \infty,$$

for some large t_*. We therefore establish that the existence interval of the solution for (8) is finite. The asymptotic profile as $x \to x_\beta$ is demonstrated in the same spirit. We may safely omit the details and we refer to [4]. The proof of Theorem 2 is thereby completed.

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Classical solutions to the stationary Navier-Stokes system in exterior domains

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1 Introduction

In a recent paper [13] we studied the Stokes system by the techniques of the potential theory and we obtained existence of solutions to the boundary value problem and a maximum modulus theorem in both cases of bounded and exterior domains, by requiring the datum to be only continuous. It is of some interest to detect whether the above results continue to hold for the Navier-Stokes system.

The boundary value problem associated with the motion of a viscous fluid filling a region exterior to a (not necessarily connected) compact set $\Omega_0 \subset \mathbb{R}^n$ and in the absence of body force, consists in finding a solution $(\mathbf{u}, \pi)$ to the Navier-Stokes system

$$\begin{aligned} \nu\Delta\mathbf{u} &= (\nabla\mathbf{u})\mathbf{u} + \nabla\pi, && \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{a} && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

As far as bounded domains are concerned, the existence and uniqueness of solutions to system (1.1) with finite Dirichlet integral has been studied for the first time in [18] (by the methods of potential theory) and subsequently by several authors (see [6,5], essentially by variational methods, for a detailed bibliography). Recently, system (1.1) with continuous boundary data has been treated in [11,12,21].

In the case of exterior domains, system (1.1) must be completed by a suitable condition at infinity. In this connection, we have to distinguish the case where the solution $\mathbf{u}$ to system (1.1) tends to a limit $\mathbf{u}_\infty \neq \mathbf{0}$ at infinity from the case where $\mathbf{u}$ tends to zero. The first problem has been studied in [8,2] (see [6,5] for a detailed bibliography). As far as we are aware, the second problem has been considered in [2] and [5] in the class of solutions vanishing at infinity as $|x|^{-n+2}$ and having a finite Dirichlet integral respectively. More precisely, in [4] it is proved that if $\mathbf{a} \in H^{\frac{1}{2}}(\partial\Omega)$,

then System (1.1) admits a unique solution $(\mathbf{u}, \varpi)$ such that $|x|\mathbf{u}(x)$ is bounded. Subsequently, [17] and [16] have been studied the behaviour of the derivatives of the field $\mathbf{u}(x)$ and in particular in [16] is proved that a solution and its derivatives have the same behaviour of a solution to the Stokes system and of the corresponding derivatives.

The aim of this note is just to show that the above results continue to hold for classical solutions. More precisely, we prove existence and uniqueness as well as a maximum modulus theorem for solutions to system (1.1) corresponding to continuous boundary data and boundary of class $C^{1,\alpha}$. Moreover we prove that our solution coincides in a suitable neighborhood of infinity with that one given in [4] and so it admits the properties established in [16].

2 Some notations and statement of the problem

In this paper, the boundary value problem (1.1) is studied in an exterior domain $\Omega = \mathbb{R}^n - \cup_{h=1}^m \bar{\Omega}_h$ where Ω_h $(h = 1, \dots, m, m \in \mathbb{N})$ are m bounded domains of $\mathbb{R}^n$ of class $C^{1,\alpha}, (\alpha \in (0,1])$ with connected boundary and such that $\bar{\Omega}_h \cap \bar{\Omega}_k = \emptyset, (h, k = 1, \dots, m, h \neq k)$.

$C^k(D)$ $(D \subseteq \mathbb{R}^n)$ is the set of scalar (as well as vector and tensor) fields on D that are continuous together with any partial derivatives up to the order k inclusive; $C^{k,\beta}(D), \beta \in (0,1]$ is the set of all fields on D that are in $C^k(D)$ and their derivatives of order k are α–hölderian on D. $W^{k,q}(D), q > 1, L^q(\Omega)$, if $k = 0, (k \in \mathbb{N}_0)$, denotes the Banach space of scalar (as well as vector and tensor) fields on D whose moduli are q–summable on D together with weak partial derivatives up to the order k inclusive; $W^{1-1/q,q}(\partial D)$ is the usual trace space of functions belonging to $W^{k,q}(D)$.

The symbol c will denote a positive constant whose value is inessential for our purposes; the numerical value of c may change from line to line and in the same line it may be, e.g., $2c \leq c$.

We shall prove the following

Theorem 2.1. *Let* $\mathbf{a} \in C(\partial\Omega)$ *and set* $A = \|\mathbf{a}\|_{C(\partial\Omega)}$. *If*

$$5B(1 + B_0)A < B_0, \quad 4AB_0B_1(1 + B_0) < 1, \tag{2.1}$$

where B, B_0 *and* B_1 *are suitable positive constants independent of* $\mathbf{a}$, *then System* (1.1) *admits a solution* $(\mathbf{u}, \varpi) \in [C^2(\Omega) \cap C(\overline{\Omega})] \times C^1(\Omega)$. *Moreover,*

$$|\mathbf{u}(x)| \leq cA|x|^{-n+2}, \quad \forall x \in \Omega; \qquad \|\mathbf{u}\|_{C(\overline{\Omega})} \leq c\|\mathbf{a}\|_{C(\partial\Omega)}. \tag{2.2}$$

Remark 2.1 - Theorem 2.1 gives an existence theorem for classical solutions corresponding to *small* data (in a suitable sense) in $C(\partial\Omega)$. This restriction is not surprising. Indeed, all the results concerning the regularity of weak solutions in $\Omega \subset \mathbb{R}^n, n \geq 4$, as well as those in the papers [2,5,17,16] are stated for small data.

Remark 2.2 - The only information about the behaviour at infinity of the solution is furnished by (2.2). As far as the behaviour of the derivatives of any order in the three dimensional case is concerned, by uniqueness we can employ the results in [16]. However, our technique allows us to prove that, for $n \geq 3$ and for any fixed k the derivatives of order less than k of the solutions to system (1.1) have the same behaviour of the derivatives of the solutions to the Stokes system. The importance of these results lies in the fact that the approximation by the Stokes system of the Navier-Stokes system can be made not only in a neighborhood of $\partial\Omega$ but also in a neighborhood of infinity (cf. [7]).

Remark 2.3 - As is well-known, the stability of the solution $(\mathbf{u}, p)$ to System (1.1) by the so-called *energy method*, is studied by detecting the behaviour of some suitable L^2-norm of the *perturbations* $\mathbf{u}$ of the kinetic field $\mathbf{v}$ which obeys the following system

$$\begin{aligned} \partial_t \mathbf{v} &= -\nabla\mathbf{v}(\mathbf{u}-\mathbf{v}) - (\nabla\mathbf{u})\mathbf{u} + \nu\Delta\mathbf{v} - \nabla\varpi, && \text{in } \Omega \times [0,+\infty), \\ \nabla \cdot \mathbf{v} &= 0, && \text{in } \Omega \times [0,+\infty), \\ \mathbf{v} &= 0, && \text{on } \partial\Omega \times [0,+\infty). \end{aligned} \tag{2.3}$$

If the functional

$$\sup_{t\geq 0} \frac{1}{\|\nabla\mathbf{w}(t)\|_2} \int_\Omega \mathbf{w}\cdot(\nabla\mathbf{v})\mathbf{w}\, dy \leq \nu^*$$

is bounded, *i. e.*, $\nu^* \in (0,+\infty)$, then, the solution $(\mathbf{u}, p)$ to System (1.1) is *unconditionally asymptotically stable*, *i. e.*,

$$\begin{aligned} &a)\ \forall\epsilon > 0, \exists\delta_\epsilon > 0 : \|\mathbf{u}(0)\|_{L^2(\Omega)} < \delta \Rightarrow \\ &\qquad \Rightarrow \|\mathbf{u}(t)\|_2 < \epsilon, \quad \forall t \geq 0, \\ &b)\ \lim_{t\to\infty} \|\mathbf{u}(t)\|_2 = 0, \end{aligned}$$

provided that

$$\nu > \nu^*. \tag{2.4}$$

The existence of ν^* and (2.4) are an easy consequence of (2.2). Stability results with respect L^2-norm and other norms of data and solutions under condition (2.4) can found in [3,14,9,10].

3 Some results about the Stokes system

We denote by $(\mathbf{U}(z), \mathbf{q}(z))$ the Stokes fundamental solution, that is, $n \geq 3$,

$$\begin{aligned} \mathbf{U}(z) &= -\frac{1}{\omega_n}\left[\frac{1}{(n-2)}\frac{\mathbf{I}}{|z|^{n-2}} + \frac{z\otimes z}{|z|^n}\right], \\ \mathbf{q}(z) &= -\frac{1}{\omega_n}\frac{z}{|z|^n}, \end{aligned}$$

where ω_n is the surface area of the unit ball of $\mathbb{R}^n$.

The following lemma are particular cases of classical results about the weakly singular kernel (cf. *e. g.* [15]):

Lemma 3.1 - *Let*

$$H(x) = \int_{\mathbb{R}^n} |x-y|^{-\mu}(1+|y|)^{-\mu_1} dy$$

with $\mu \in [0,n)$ *and* $n \neq \mu_1 > 0$ *and such that* $\mu + \mu_1 > n$. *Then*

$$H(x) \leq c/(1+|x|)^{\overline{\mu}}, \forall x \in \mathbb{R}^n, \text{ with } \overline{\mu} = \min\{\mu, \mu+\mu_1 - n\},$$

where c is a constant independent of x.

Lemma 3.2 - *Let*

$$T(\Phi)(x) = \int_\Omega \nabla \mathbf{U}(x-y) \cdot \Phi(y) dy, \forall x \in \Omega,$$

with Φ *tensor field, then*

i) if $\Phi(y) \in L^p(\Omega) \cap L^q(\Omega), p > n, q < n, \to T(\Phi)(x) \in C^{0,\beta}(\overline{\Omega}), \forall \beta \in (0, n/p)$ *with* $|T(\Phi)|_{C^{0,\beta}} \leq c[|\Phi|_p + |\Phi|_q]$;

ii) if $\Phi(y) \in C^{0,\beta}(\Omega') \cap L^q(\Omega)$, *for some* $q > 1 \to T(\Phi)(x) \in C^{1,\beta'}(\Omega'), \forall \beta' \in (0, \min\{\alpha, \beta\}), \forall \Omega' \subseteq \Omega$ *with* $|T(\Phi)|_{C^{1,\beta'}(\Omega')} \leq c(\Omega')[|\Phi|_q^{1-\beta'/\beta}[\Phi]_{0,\beta}^{\beta'/\beta} + [\Phi]_{0,\beta}]$, *with* $[\cdot]_{0,\beta}$ *Hölder coefficient.*

Lemma 3.3 - *Let*

$$\tilde{T}(\Phi)(x) = \int_{\partial\Omega} \mathbf{U}(x-\eta) \cdot \Phi(\eta) d\sigma_\eta, \forall x \in \Omega.$$

Then $\tilde{T}$ *is completely continuous from* $C(\partial\Omega)$ *into* $C^{0,\beta}(\overline{\Omega})$, *with* $\beta < 1$. *Moreover* $\tilde{T}(\Phi)(x) \in C^\infty(\Omega)$.

Let us consider the Stokes system

$$\begin{aligned} &\Delta \mathbf{u}(x) - \nabla\pi(x) = \mathbf{f}(x), \ \nabla \cdot \mathbf{u}(x) = 0, \text{on } \Omega, \\ &\mathbf{u}(x)_{|\partial\Omega} = \mathbf{a}(x), \ \text{with} \int_{\partial\Omega} \mathbf{a}(\xi) \cdot \vec{n} d\sigma_y = 0 \text{ if } \Omega \text{ bounded}, \\ &\mathbf{u}(x) \to 0 \text{ for } |x| \to \infty \text{ if } \Omega \text{ exterior}. \end{aligned} \tag{3.1}$$

The following theorems can be found in [13]

Theorem 3.1 - *Let* Ω *be an exterior domain of class* $C^{1,\alpha}$ *and let* $\mathbf{a} \in C(\partial\Omega)$. *Then there exists a classical solution* $(\mathbf{u}, p) \in C(\overline{\Omega}) \cap C^2(\Omega)$ *to System* (3.1), *with*

$\mathbf{f} = 0$, *which, for* $n \geq 3$, *is unique in the class* $\{(\mathbf{u}, p) |\ \mathbf{u} = o(1)\}$. *Moreover, there exists a constant* c *independent of* $\mathbf{a}$ *such that*

$$|\mathbf{u}(x)| \leq c(1 + |x|)^{-n+2}. \qquad 3.2$$

If $\mathbf{a} \in C^{0,\beta}(\partial\Omega)$, *with* $\beta \leq \alpha$, *then* $\mathbf{u} \in C^{0,\beta}(\overline{\Omega})$ *and*

$$|\mathbf{u}|_{C^{0,\beta}(\overline{\Omega})} \leq c|\mathbf{a}|_{C^{0,\beta}(\partial\Omega)}. \qquad 3.3$$

Theorem 3.2 - *Let* D *be bounded of class* $C^{k,(2-k)\alpha}$, *let* $\mathbf{a} \in W^{k-1/q,q}(\partial D)$ *satisfying* $(3.1)_4$, *and* $\mathbf{f} \in W^{k-2,q}(D)$, *with* $k = 1, 2$. *Then there exist a unique weak solution* $\mathbf{u} \in W^{k,q}(D)$ *to system* (3.1), *with an associated pressure field* $p \in W^{k-1,q}(D)$, *such that*

$$||\mathbf{u}||_{W^{k,q}(D)} + ||p||_{W^{k-1,q}(D)} \leq c(||\mathbf{a}||_{W^{k-1/q,q}(\partial D)} + |\mathbf{f}|_{W^{k-2,q}}),$$

with c *independent of* $\mathbf{a}$ *and* $\mathbf{f}$.

If D *is an exterior domain, then the same results hold for* $q \in (1, n/(n-2))$ *if* $\mathbf{f} \in W^{-1,q}(D)$ *and for* $q \in (1, n/(n-1))$ *if* $\mathbf{f} \in L^q(D)$.

Theorem 3.3 - *Let* D *be bounded. Let* $\mathbf{u} \in \overset{\circ}{W}{}^{1,q}(D)$ *be a weak solution to system* (3.1) *with homogeneous data. Then* $\mathbf{u} = \mathbf{0}$ *in* D.

Theorem 3.4 - *Let* D *be bounded. Let* $\mathbf{f} \in C^{0,\beta}(D)$ *and let* $\mathbf{a} \in C(\partial D)$ *satisfy* $(3.1)_4$. *Then system (3.1) has a unique classical solution* $(\mathbf{u}, p)$ *and*

$$\max_{\overline{D}} |\mathbf{u}(x)| \leq c(\max_{\partial D} |\mathbf{a}(\xi)| + \max_{\overline{D}} |\mathbf{f}|),$$
$$\sum_{|k| \leq 2} |D^k \mathbf{u}(x)| + \sum_{|k| \leq 1} |D^k p(x)| \leq c(K)(\max_{\partial D} |\mathbf{a}(\xi)| + |\mathbf{f}|_{C^{0,\beta}}), \forall x \in K,$$
$$\forall K \subset D \text{ such that } dist(K, \partial D) > 0.$$

Corollary 3.1 - *Let* $\mathbf{u}$ *be a weak solution to system* (3.1), *with* $\mathbf{f} = 0$. *Then, we can associate a pressure field* p *to* $\mathbf{u}$ *in such a way that* $(\mathbf{u}, p) \in C^\infty(\Omega) \times C^\infty(\Omega)$ *is a solution to system* (3.1).

Remark 3.1 -

Of course, if $\mathbf{f} \neq 0$, then regularity of the solution to System 3.1 depends on the regularity of $\mathbf{f}$.

These results reproduce some classical ones in [1,19,20].

4 Proof of Theorem 2.1

Throughout the proof of the theorem we denote by B_0 a constant greater than the ones appearing in Theorems 3.1 - 3.3 and by B_1 a constant greater than the ones appearing in Lemmas 3.1 - 3.3.

In virtue of the results of Section 3, there exist a solution $\mathbf{u}_0$ to System (3.1), with $\mathbf{f} = 0$, and a positive constant B_0, independent of $\mathbf{a}$, such that

$$|\mathbf{u}_0(x)| \leq B_0 A|x|^{-1}, \qquad |\mathbf{u}_0(x)| \leq B_0 A, \quad \forall x \in \overline{\Omega}. \tag{4.1}$$

Let us consider the boundary value problem

$$\begin{aligned} \Delta \mathbf{u}_n - \nabla \varpi_n &= (\nabla \mathbf{u}_{n-1})\mathbf{u}_{n-1} \\ \nabla \cdot \mathbf{u}_n &= 0, \\ \mathbf{u}_n &= \mathbf{a}, \\ \mathbf{u}_n &= o(1), \end{aligned}$$

Let us show that System (4.2) defines a sequence of fields $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that

$$\begin{aligned} &\mathbf{u}_n \in C(\overline{\Omega}) \cap C^2(\Omega) \\ &|\mathbf{u}_n(x)| \leq 2B_0 A|x|^{-1}, \quad |\mathbf{u}_n(x)| \leq 2B_0 A, \quad \forall x \in \Omega, \\ &|\mathbf{u}_n|_{C^{0,\beta}(\Omega')} \leq AB_0(B_1 A + C(\Omega')), \\ &|\mathbf{u}_n - \mathbf{u}_0|_{C^{0,\beta}(\Omega)} \leq (2B)^3 A^2, \end{aligned} \tag{4.3}$$

with B_0, B_1, B and $C(\Omega')$ independent of n.

Of course, the properties (4.3) are true for $n = 0$. Then, in order to get the desired result by induction, we have to prove that, if $\mathbf{u}_{n-1}$ satisfies (4.3), then so does $\mathbf{u}_n$. To this end, set

$$\mathbf{u}_n = \mathbf{v}_n + \tilde{\mathbf{u}}_n + \mathbf{u}_0,$$

where $\mathbf{v}_n$ is a solution to the system

$$\begin{aligned} \Delta \mathbf{v}_n - \nabla p_n &= (\nabla \hat{\mathbf{u}}_{n-1})\hat{\mathbf{u}}_{n-1}, && \text{in } \mathbb{R}^n \\ \nabla \cdot \mathbf{v}_n &= 0, && \text{in } \mathbb{R}^n \\ \mathbf{v}_n &= o(1), && \text{as } |x| \to \infty, \end{aligned} \tag{4.4}$$

with

$$\hat{\mathbf{u}}_{n-1}(x) = \begin{cases} \mathbf{u}_{n-1}(x), & x \in \overline{\Omega}, \\ \mathbf{0}, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and $\tilde{\mathbf{u}}_n$ is a solution to the Stokes system

$$\begin{aligned} \Delta \tilde{\mathbf{u}}_n - \nabla \tilde{\pi}_n &= \mathbf{0}, && \text{in } \Omega, \\ \nabla \cdot \tilde{\mathbf{u}}_n &= 0, && \text{in } \Omega, \\ \tilde{\mathbf{u}}_n &= -\mathbf{v}_n, && \text{on } \partial\Omega, \\ \tilde{\mathbf{u}}_n &= o(1), && \text{as } |x| \to \infty. \end{aligned} \tag{4.5}$$

Next, by taking into account that $\mathcal{I}_2$ is a single layer potential with a smooth density, we have that there exists a positive constant B_2 such that

$$|\mathcal{I}_2(x)| \leq B_2 A^2 |x|^{-n+2} \quad \text{and} \quad |\mathcal{I}_2(x)| \leq B_2 A^2, \quad \forall x \in \mathbb{R}^n.$$

Therefore, setting $B = \max\{B_1, B_2, B_0^2\}$, we obtain

$$|\mathbf{v}_n(x)| \leq 5BA^2|x|^{-n+2} \text{ and } |\mathbf{v}_n(x)| \leq 5BA^2,\ \forall x \in \mathbb{R}^n,\ \forall n \in \mathbb{N}. \tag{4.6}$$

Consider now the solution $\tilde{\mathbf{u}}_n$ to System (4.4). Taking into account the estimates on $\mathbf{v}_n$, in virtue of Theorem 3.1, we see that

$$|\tilde{\mathbf{u}}_n(x)| \leq 5BB_0A^2|x|^{-n+2} \quad \text{and} \quad |\tilde{\mathbf{u}}_n(x)| \leq 5BB_0A^2, \quad \forall x \in \mathbb{R}^n,\ \forall n \in \mathbb{N},$$

and $\tilde{\mathbf{u}}_n \in C^{0,\beta}(\overline{\Omega}) \cap C^2(\Omega)$, with

$$|\tilde{\mathbf{u}}_n|_{C^{0,\beta}(\overline{\Omega})} \leq B_0 |\mathbf{v}_n|_{C^{0,\beta}(\partial\Omega)} \leq 4B_0B^2A^2.$$

Therefore, bearing in mind that $\mathbf{u}_n = \mathbf{v}_n + \tilde{\mathbf{u}}_n + \mathbf{u}_0$, we get

$$\left.\begin{aligned} |\mathbf{u}_n(x)| &\leq [5B(1+B_0)A^2 + B_0A]|x|^{-n+2}, \\ |\mathbf{u}_n(x)| &\leq 5B(1+B_0)A^2 + B_0A, \end{aligned}\right. \qquad \forall x \in \overline{\Omega}$$

and estimate $(4.3)_3$ for the Hölder norm. Since, by hypothesis, $5B(1+B_0)A < B_0$, we have proved that $\mathbf{u}_n$ satisfies (4.3), for any $n \in \mathbb{N}$, with the exception of C^2-regularity. On the other hand, since the regularity of the solutions is a local property, it is a simple consequence of the Theorem 3.4 applied to System 4.2.

Let us show that the sequence $\{\mathbf{u}_n(x)\}_{n\in\mathbb{N}}$ just constructed, converges, as $n \to \infty$, to a solution to the Navier–Stokes system (1.1). To this end set

$$\mathbf{u}_n(x) = \sum_{k=1}^{n} [\mathbf{u}_k(x) - \mathbf{u}_{k-1}(x)] + \mathbf{u}_0(x) = \sum_{k=1}^{n} \mathbf{w}_k(x) + \mathbf{u}_0(x).$$

For any $k \in \mathbb{N}$, we have that

$$\begin{aligned} \Delta \mathbf{w}_{k+1} - \nabla p_{k+1} &= (\nabla \mathbf{u}_k)\mathbf{w}_k + (\nabla \mathbf{w}_k)\mathbf{u}_{k-1}, && \text{in } \Omega, \\ \nabla \cdot \mathbf{w}_{k+1} &= 0, && \text{in } \Omega \\ \mathbf{w}_{k+1} &= \mathbf{0}, && \text{on } \Omega \\ \mathbf{w}_{k+1} &= o(1), && \text{as } |x| \to \infty \end{aligned}$$

Let us show that the sequence $\{\mathbf{u}_n(x)\}_{n\in\mathbb{N}}$ just constructed, converges, as $n \to \infty$, to a solution to the Navier–Stokes system (1.1). To this end set

$$\mathbf{u}_n(x) = \sum_{k=1}^{n} [\mathbf{u}_k(x) - \mathbf{u}_{k-1}(x)] + \mathbf{u}_0(x) = \sum_{k=1}^{n} \mathbf{w}_k(x) + \mathbf{u}_0(x).$$

For any $k \in \mathbb{N}$, we have that

$$\begin{aligned} \Delta\mathbf{w}_{k+1} - \nabla p_{k+1} &= (\nabla\mathbf{u}_k)\mathbf{w}_k + (\nabla\mathbf{w}_k)\mathbf{u}_{k-1}, && \text{in } \Omega,\\ \nabla\cdot\mathbf{w}_{k+1} &= 0, && \text{in } \Omega\\ \mathbf{w}_{k+1} &= \mathbf{0}, && \text{on } \Omega\\ \mathbf{w}_{k+1} &= o(1), && \text{as } |x| \to \infty \end{aligned}$$

and

$$\begin{aligned} &\mathbf{w}_k \in C(\overline{\Omega}) \cap C^2(\Omega),\\ &|\mathbf{w}_k(x)| \le (4AB_0B_1)^{k+1}(1+B_0)^k|x|^{-n+2},\\ &|\mathbf{w}_k(x)| \le (4AB_0B_1)^{k+1}(1+B_0)^k, \quad \forall x \in \overline{\Omega},\\ &\mathbf{w}_k \in C^{1,\beta'}(\overline{\Omega}) \cap C^{0,\beta}(\overline{\Omega}), \text{ with}\\ &|\mathbf{w}_k|_{C^{0,\beta}(\overline{\Omega})} \le (2B)^3(4AB_0B_1)^{k+1}(1+B_0)^k \text{ and}\\ &|\mathbf{w}_k|_{C^{1,\beta'}(\overline{\Omega})} \le (2B)^3(4AB_0B_1)^{k+1}(1+B_0)^k. \end{aligned} \tag{4.7}$$

Relation $(4.7)_1$ is an immediate consequence of the definition of $\mathbf{w}_k(x)$. In order to prove $(4.7)_2$, set

$$\mathbf{w}_{k+1}(x) = \omega_{k+1}(x) + \tilde{\mathbf{w}}_{k+1}(x),$$

where ω_k is a solution to the System

$$\begin{aligned} \Delta\omega_{k+1} - \nabla\hat{p}_{k+1} &= (\nabla\mathbf{u}_k)\hat{\mathbf{w}}_k + (\nabla\hat{\mathbf{w}}_k)\mathbf{u}_{k-1}, && \text{in } \mathbb{R}^n\\ \nabla\cdot\omega_{k+1} &= 0 && \text{in } \mathbb{R}^n,\\ \omega_{k+1} &= o(1) && \text{as } |x| \to \infty, \end{aligned}$$

with $\hat{\mathbf{w}}_k \in C^{0,\beta}(\mathbb{R}^n)$, $\hat{\mathbf{w}}_k = \mathbf{w}_k$ in $\overline{\Omega}$ and $\hat{\mathbf{w}}_k = \mathbf{0}$ on $\mathbb{R}^n \setminus \Omega$, and $\tilde{\mathbf{w}}_{k+1}$ is a classical solution to the Stokes system

$$\begin{aligned} \Delta\tilde{\mathbf{w}}_{k+1} - \nabla\tilde{p}_{k+1} &= 0, && \text{in } \Omega,\\ \nabla\cdot\tilde{\mathbf{w}}_{k+1} &= 0, && \text{in } \Omega,\\ \tilde{\mathbf{w}}_{k+1} &= -\omega_{k+1}, && \text{on } \partial\Omega,\\ \tilde{\mathbf{w}}_{k+1} &= o(1), && \text{as } |x| \to \infty. \end{aligned}$$

The solution ω_{k+1} can be written as

$$\begin{aligned} \omega_{k+1}(x) &= \int_{\mathbb{R}^n} \mathbf{U}(x-y)\,[(\nabla\mathbf{u}_k)\hat{\mathbf{w}}_k + (\nabla\hat{\mathbf{w}}_k)\mathbf{u}_{k-1}]\,dv_y\\ &= -\int_{\mathbb{R}^n} [\nabla\mathbf{U}(x-y)](\hat{\mathbf{w}}_k \otimes \mathbf{u}_{k-1} + \mathbf{u}_k \otimes \hat{\mathbf{w}}_k)dv_y. \end{aligned} \tag{4.8}$$

In virtue of (4.8), (4.3) and Lemmas 3.1-3.2, it is simple to realize that, if

$$|\widehat{\mathbf{w}}_k(x)| \le (4AB_0B_1)^k(1+B_0)^{k-1}|x|^{-n+2},$$
$$|\widehat{\mathbf{w}}_k(x)| \le (4AB_0B_1)^k(1+B_0)^{k-1}, \qquad \forall x \in \mathbb{R}^n$$
$$|\widehat{\mathbf{w}}_k(x)|_{C^{0,\beta}(\mathbb{R}^n)} \le (2B)^3(4AB_0B_1)^{k+1}(1+B_0)^{k-1},$$

then

$$|\omega_{k+1}(x)| \le (4AB_0B_1)^{k+1}(1+B_0)^{k-1}|x|^{-n+2},$$
$$|\omega_{k+1}(x)| \le (4AB_0B_1)^{k+1}(1+B_0)^{k-1}, \qquad \forall x \in \mathbb{R}^n \tag{4.9}$$
$$|\omega_{k+1}(x)|_{C^{1,\beta}(\mathbb{R}^n)} \le (2B)^3(4AB_0B_1)^{k+1}(1+B_0)^{k-1}.$$

Consider now the field $\tilde{\mathbf{w}}_{k+1}(x)$. In virtue of Theorem 3.1, it satisfies inequalities (4.9):

$$|\tilde{\mathbf{w}}_{k+1}(x)| \le B_0(4AB_0B_1)^{k+1}(1+B_0)^{k-1}|x|^{-n+2},$$
$$|\tilde{\mathbf{w}}_{k+1}(x)| \le B_0(4AB_0B_1)^{k+1}(1+B_0)^{k-1}, \ \forall x \in \overline{\Omega}$$
$$|\tilde{\mathbf{w}}_{k+1}(x)|_{C^{1,\beta'}(\overline{\Omega})} \le (2B)^4(4AB_0B_1)^{k+1}(1+B_0)^{k-1} \text{ and}$$
$$|\tilde{\mathbf{w}}_{k+1}|_{C^{0,\beta}(\overline{\Omega})} \le (2B)^3(4AB_0B_1)^{k+1}(1+B_0)^{k-1}.$$

Hence, bearing in mind the expression of $\mathbf{w}_k$, we arrive at (4.8). Putting together the above results, we have that the sequence $\{\mathbf{u}_n(x)\}_{n\in\mathbb{N}}$ converges to a limit $\mathbf{u}(x) \in C(\overline{\Omega})$ such that

$$\mathbf{u}(x) = \omega(x) + \tilde{\mathbf{w}}(x) + \mathbf{u}_0(x), \quad \forall x \in \overline{\Omega}$$
$$\mathbf{u}(x) = \mathbf{a}(x), \qquad \forall x \in \partial\Omega,$$

with

$$\omega, \tilde{\mathbf{w}} \in C(\overline{\Omega}) \cap C^{0,\beta}(\overline{\Omega}) \cap C^{1,\beta'}(\Omega),$$
$$\mathbf{u}_0(x) \in C(\overline{\Omega}) \cap C^2(\Omega), \tag{4.10}$$
$$\mathbf{u} \le M4AB_0B_1(1+B_0)|x|^{-n+2}, \ |\mathbf{u}(x)| \le M4AB_0B_1(1+B_0), \ \forall x \in \overline{\Omega},$$

and

$$M = \frac{1}{1-4AB_0B_1(1+B_0)}.$$

The field $\mathbf{u}(x)$ is in particular a weak solution to the Navier–Stokes system with $\nabla\mathbf{u} \in L^2_{loc}(D), \overline{D} \subset \Omega$, so that, in virtue of Theorems 3.2 -3.3 and Corollary 3.1 we have that $\mathbf{u} \in C^2(\Omega)$ and there exists a field $p \in C^1(\Omega)$ such that the couple $(\mathbf{u}, p)$ is a regular solution in Ω to System (1.1).

The existence of a classical solution is completely proved. To prove uniqueness in the class of solutions $C(\overline{\Omega})$ that behave at infinity as $|x|^{-n+2}$, we observe that the difference $\mathbf{v}$ of two solutions satisfies the system

$$\begin{aligned}
\nabla\mathbf{v}(\mathbf{u}+\mathbf{v}) + (\nabla\mathbf{u})\mathbf{u} &= \nu\Delta\mathbf{v} - \nabla\varpi, && \text{in } \Omega,\\
\nabla\cdot\mathbf{v} &= 0, && \text{in } \Omega,\\
\mathbf{v} &= 0, && \text{on } \partial\Omega.\\
\mathbf{v}(x) &\to 0, && |x| \to 0.
\end{aligned} \tag{4.11}$$

and that thanks to property (2.2) and the smallness of data, uniqueness is a simple consequence of the *energy relation* deduced from system (4.11). In order to prove the energy relation we must prove that $\nabla \mathbf{v} \in L^2(\Omega)$. Set $\mathbf{f} = (\nabla \mathbf{v})(\mathbf{u}+\mathbf{v}) + (\nabla \mathbf{u})\mathbf{u}$. Since $\mathbf{v}\otimes\mathbf{v}$, $\mathbf{v}\otimes\mathbf{u} = O(|x|^{-2n+4})$, by Theorem 3.2 there exists a weak solution $(\mathbf{w}, p)$ to the system

$$\begin{aligned} \nu\Delta\mathbf{v} - \nabla\varpi &= \mathbf{f}, && \text{in } \Omega, \\ \nabla\cdot\mathbf{v} &= 0, && \text{in } \Omega, \\ \mathbf{v} &= 0, && \text{on } \partial\Omega. \\ \mathbf{v}(x) &\to 0, && |x| \to 0, \end{aligned}$$

such that $\nabla\mathbf{w} \in L^2(\Omega)$. In virtue of the regularity of $\mathbf{f}$, we have that $\mathbf{w}$ is sufficiently smooth in Ω so that by Theorems 3.2 - 3.3 it is not difficult to deduce that $\nabla\mathbf{w} \in L^q(\Omega\cap S_R), \forall q > 1$. Thus $\mathbf{w} \in C(\overline{\Omega})\cap C^2(\Omega)$. By the uniqueness of classical solutions to Stokes problem in exterior domains [13], we have that $\mathbf{v}$ and $\mathbf{w}$ coincide so that $\nabla\mathbf{v} \in L^2(\Omega)$. Thus the uniqueness is achieved.

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Stationary Navier-Stokes flow in 2-dimensional Y-shape channel under general outflow condition

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Abstract

We consider the stationary Navier-Stokes equations in a 2-dimensional Y-shape infinite channel involving the general outflow condition. Assuming that the domain and the data are symmetric with respect to a straight line, and that the line of symmetry intersects every boundary component where the boundary integral does not vanish, we show the existence of symmetric solutions. We report also the regularity and the asymptotic behavior of the solution.

Keywords: Stationary Navier-Stokes equations, Non-vanishing outflow, 2 dimensional Y-shape channel, Symmetry
1991 Mathematics Subject Classification: 35Q30, 76D05.

1 Introduction

Let Ω be a two dimensional domain with unbounded boundary, symmetric with respect to the x_2-axis satisfying the following conditions.

(A-1) For some large $R_0 > 0$ and $0 < \alpha < \pi$, Ω consists of four parts.

$$\Omega_1 := \{(x_1, x_2)\ ;\ x_1 \sin\alpha + x_2 \cos\alpha > R_0,\ |x_1 \cos\alpha - x_2 \sin\alpha| < 1\}$$
$$\Omega_2 := \{(x_1, x_2)\ ;\ -x_1 \sin\alpha + x_2 \cos\alpha > R_0,\ |x_1 \cos\alpha + x_2 \sin\alpha| < 1\}$$
$$\Omega_3 := \{(x_1, x_2)\ ;\ |x_1| < 1,\ x_2 < -R_0\} \text{ and}$$
$$\Omega \setminus (\Omega_1 \cup \Omega_2 \cup \Omega_3) \text{ is bounded.}$$

Put $\Omega_0 := \Omega \setminus \overline{\Omega_1 \cup \Omega_2 \cup \Omega_3}$.

(A-2) The boundary $\partial\Omega$ is smooth and consists of infinite part γ_0, and finite part γ_1, γ_2, $\cdots$, γ_N the latter being simply closed curves. The infinite part γ_0 consists of three curves $\gamma_0^i (i = 1, 2, 3)$, where γ_0^1 is the upper part, γ_0^2 the left part, and γ_0^3 the right part. Namely, $\partial\Omega = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \cdots \gamma_N$. Furthermore, γ_0^1, γ_1, $\cdots$, γ_N intersect the x_2-axis.

We call the domain Ω satisfying these conditions Y-shape channel, because it looks like the letter Y for $0 < \alpha < \pi/2$. Let $T = \{(x_1, x_2) \ ; \ |x_1| < 1\}$ and $\boldsymbol{U}(x_1, x_2) = \frac{3}{4}(0, 1 - x_1^2)$. T_1 is the rotation of T by the angle $-\alpha$ around the origin, and T_2, by the angle α. $\boldsymbol{U}_1$ (resp. $\boldsymbol{U}_2$) is the standard Poiseuille flow in T_1 (resp. T_2).

$$T_1 = \{(x_1, x_2) \ ; \ |x_1 \cos\alpha - x_2 \sin\alpha| < 1\}$$

$$T_2 = \{(x_1, x_2) \ ; \ |x_1 \cos\alpha + x_2 \sin\alpha| < 1\}$$

$$\boldsymbol{U}_1 = \frac{3}{4}\{1 - (x_1 \cos\alpha - x_2 \sin\alpha)^2\}\boldsymbol{e}, \quad \boldsymbol{e} = (\sin\alpha, \cos\alpha)$$

$$\boldsymbol{U}_2 = \frac{3}{4}\{1 - (x_1 \cos\alpha + x_2 \sin\alpha)^2\}\boldsymbol{f}, \quad \boldsymbol{f} = (-\sin\alpha, \cos\alpha)$$

We study the Navier-Stkes flow in the Y-shape infinite channel such that the velosity tends to Poiseuille flow at the infinity, and there are some inflow and outflow from the boundary. More exactly, we consider the boundary value problem for the Navier-Stokes equations.

$$(NS)\left\{\begin{array}{rcll} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} & = & \nu\Delta\boldsymbol{u} - \nabla p & \text{in } \ \Omega, \\ \operatorname{div}\boldsymbol{u} & = & 0 & \text{in } \ \Omega, \end{array}\right.$$

$$(BC)\left\{\begin{array}{lllll} \boldsymbol{u} = & \boldsymbol{\beta} & & & \text{on } \partial\Omega, \\ \boldsymbol{u} \to & \mu\boldsymbol{U}_1 & \text{as } x_1 \to & +\infty & \text{in } \ \Omega, \\ \boldsymbol{u} \to & \mu\boldsymbol{U}_2 & \text{as } x_1 \to & -\infty & \text{in } \ \Omega, \\ \boldsymbol{u} \to & \lambda\boldsymbol{U} & \text{as } x_2 \to & -\infty & \text{in } \ \Omega, \end{array}\right.$$

where $\boldsymbol{u}$ is the velosity of the fluid, p is the pressure, λ, μ are real numbers, $\boldsymbol{\beta}$ is a given function on the boundary $\partial\Omega$, smooth, symmetric with respect to the x_2-axis satisfying

(A-3) $\boldsymbol{\beta} = 0$ on γ_0 and $\displaystyle\sum_{i=1}^{N} \int_{\gamma_i} \boldsymbol{\beta}\cdot\boldsymbol{n}\, d\sigma + 2\mu - \lambda = 0,$

$\boldsymbol{n}$ being the unit outward normal vector to $\partial\Omega$. The vector field $\boldsymbol{u} = (u_1, u_2)$ is called symmetric with respect to the x_2-axis if $u_1(x_1, x_2)$ is an odd function of x_1 and $u_2(x_1, x_2)$ is an even function of x_1, *i.e.*,

$$u_1(-x_1, x_2) = -u_1(x_1, x_2), \quad u_2(-x_1, x_2) = u_2(x_1, x_2)$$

holds true.

The purpose of this report is to prove the existence of a symmetric solution to (NS)(BC), the regularity and the asymptotic behavior of the solution.

We note that the number of curves which form the inner boundary is arbitrarily finite, and that the angle α is also arbitrary.

For "full" infinite channel and semi-infinite channel, symmetric with respect to the axis, we have shown the existence of symmetric solutions. See Morimoto-Fujita[3], [4]. We have also reported the existence result for V-shape channel [5].

In Section 2, we state the notation and the results: existence of symmetric solution(Theorem 1) and its regularity(Theorem 2). In Section 3, we construct a solenoidal extension of the boundary value under general outflow condition, and "connect" three Poiseuille flows in Ω. In Section 4, we give a proof of Theorem 1. Proof of Theorem 2 is similar to that in Morimoto-Fujita[3] and is omitted.

2 Notation and results

We need the following function spaces. Let $L^2(\Omega)$ be the set of all $\mathbf{R}^2$-valued square integrable functions defined in Ω with the inner product $(\cdot, \cdot)_\Omega$ and the norm $||\cdot||_\Omega$. If it is clear from the context, we write simply $(\cdot, \cdot)$ and $||\cdot||$. Let $C_{0,\sigma}^\infty(\Omega)$ be the set of all smooth solenoidal $\mathbf{R}^2$-valued functions with compact support in Ω. $H_\sigma = H_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)$, and $V = V(\Omega)$ is the completion of $C_{0,\sigma}^\infty(\Omega)$ in the Dirichlet norm $||\nabla\cdot||$. Let $C_{0,\sigma}^{\infty,S}(\Omega)$ be the set of all functions in $C_{0,\sigma}^\infty(\Omega)$ symmetric with respect to the x_2-axis, and $V^S = V^S(\Omega)$ the completion of $C_{0,\sigma}^{\infty,S}(\Omega)$ in the Dirichlet norm $||\nabla\cdot||$.

Let $\mathcal{P}$ be the set of all functions $\boldsymbol{W}$ which is symmetric with respect to the x_2-axis, solenoidal in Ω, belongs to $H^1(\omega)$ for every bounded open subset ω of Ω, and

$$\boldsymbol{W} = \begin{cases} \mu \boldsymbol{U}_1 & \text{in} \quad \Omega_1 \\ \mu \boldsymbol{U}_2 & \text{in} \quad \Omega_2 \\ \lambda \boldsymbol{U} & \text{in} \quad \Omega_3. \end{cases}$$

We denote by $\mathcal{S}_\sigma^{\beta,\mathcal{P}}$ the set of all functions of the form $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{W} + \boldsymbol{b}_0$ where $\boldsymbol{w}$ is in $V(\Omega)$, $\boldsymbol{W}$ is in $\mathcal{P}$ and $\boldsymbol{b}_0$ is a solenoidal extension of $\boldsymbol{\beta} - \boldsymbol{W}|_{\partial\Omega}$ with compact support.

We consider the problem to find $\boldsymbol{u} \in \mathcal{S}_\sigma^{\beta,\mathcal{P}}$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) = 0 \quad (\forall \boldsymbol{v} \in C_{0,\sigma}^{\infty,S}(\Omega)) \tag{0.1}$$

holds true.

Let

$$\sigma := \sup_{\varphi \in V(T)} \frac{((\varphi\cdot\nabla)\varphi, \boldsymbol{U})}{||\nabla\varphi||^2}, \tag{0.2}$$

$$\sigma^S := \sup_{\varphi \in V^S(T)} \frac{((\varphi\cdot\nabla)\varphi, \boldsymbol{U})}{||\nabla\varphi||^2}. \tag{0.3}$$

It is known that $0 < \sigma^S \leq \sigma$. See [4]. Our main result is as follows.

Theorem 1. *Let Ω be a domain symmetric with respect to the x_2-axis satisfying the condition (A-1), (A-2) and β be a smooth function defined on $\partial\Omega$, symmetric with respect to the x_2-axis vanishing on γ_0 and satisfying the condition (A-3). If $\nu > \max(|\mu|\sigma, |\lambda|\sigma^S)$, then there exists $u \in S_\sigma^{\beta,\mathcal{P}}$ which is symmetric with respect to the x_2-axis and satisfies (0.1).*

Theorem 2. *The solution u obtained in Theorem 1 is continuous in Ω_1 (resp. Ω_2, Ω_3), and converges to the Poiseuille flow μU_1 (resp. μU_2, λU) uniformly at the infinity.*

Remark 1. *We can replace (A-3) by the following condition:*

(A-3)' *the support of β is compact, $\int_{\gamma_0^i} \beta \cdot n = 0$ for $i = 2, 3$ and*

$$\int_{\gamma_0^1} \beta \cdot n d\sigma + \sum_{i=1}^{N} \int_{\gamma_i} \beta \cdot n d\sigma + 2\mu - \lambda = 0.$$

We remark that every integral $\int_{\gamma_i} \beta \cdot n d\sigma$ need not to vanish nor to be small.

Remark 2. *If we may say that Y-shape channel has two "hands", our result holds true for the domain having more than two hands.*

3 Preliminaries

Firstly, we begin with the well known result about the solenoidal extension of the boundary value.

Lemma 1. *Let β_0 be defined on $\partial\Omega$, smooth and have a compact support. Suppose*

$$(SOC) \int_{\gamma_i} \beta_0 \cdot n d\sigma = 0 \ (i = 1, 2, \cdots, N), \quad \int_{\gamma_0^i} \beta_0 \cdot n d\sigma = 0 \ (i = 1, 2, 3).$$

Then for every $\varepsilon > 0$ there exists a solenoidal extension of β_0 of compact support satisfying

$$|((v \cdot \nabla)v, b_0)| \leq \varepsilon ||\nabla v||^2 \quad (\forall v \in V(\Omega)).$$

If β_0 does not enjoy the condition (SOC), we can not expect the existence of its solenoidal extension satisfying the above inequality. Nevertherless, if it is symmetric, we have

Lemma 2. *Let β_0 be smooth on $\partial\Omega$, symmetric with respect to the x_2-axis and have compact support. Suppose*

$$\int_{\partial\Omega} \beta_0 \cdot n d\sigma = \sum_{i=1}^{N} \int_{\gamma_i} \beta_0 \cdot n d\sigma + \int_{\gamma_0^1} \beta_0 \cdot n d\sigma = 0, \quad \int_{\gamma_0^i} \beta_0 \cdot n d\sigma = 0 \ \ (i = 2, 3).$$

Then, for every positive ε, there exists a solenoidal symmetric extension b_0 of β_0, such that

$$|((v \cdot \nabla)v, b_0)| \leq \varepsilon ||\nabla v||^2 \quad (\forall v \in V^S(\Omega))$$

holds true.

In order to prove this Lemma, we use the virtual drain method originated by Fujita[2]. For the detail, see Fujita[2]. Remark that γ_0^2 and γ_0^3 do not intersect the x_2-axis.

Let us construct an element of $\mathcal{P}$, that is, let us 'Mconnect" three Poiseuille flows μU_1, μU_2 and λU in Ω. Let $\Sigma_i = \partial\Omega_i \cap \partial\Omega_0 (i = 1, 2, 3)$. Consider the following Stokes problem.

$$-\Delta v + \nabla q = 0, \ \operatorname{div} v = 0 \ \text{ in } \ \Omega_0$$

$$v|_{\Sigma_1} = \mu U_1, \quad v|_{\Sigma_2} = \mu U_2, \quad v|_{\Sigma_3} = \lambda U$$

$$v|_{\gamma_i} = 0 \ (i = 1, \cdots, N),$$

$$v|_{\gamma_0^1 \cap \partial\Omega_0} = (0, (\lambda - 2\mu)\eta(x_1)), \quad v|_{\gamma_0^i \cap \partial\Omega_0} = 0 \ (i = 2, 3),$$

where $\eta(t)$ is a smooth even function with small compact support, the integral $\int_{-\infty}^{\infty} \eta(t)dt = 1$. Let v_0 be the solution in $H^1(\Omega_0)$. v_0 is symmetric with respect to the x_2-axis, because the boundary value is symmetric with respect to the x_2-axis. Since v_0 satisfies the stringent outflow condition, that is, for every connected component of the boundry of Ω_0 the integral $\int v_0 \cdot n d\sigma = 0$, there exists its stream function φ_0, and $v_0 = \nabla^\perp \varphi_0 = (-D_2\varphi_0, D_1\varphi_0)$ holds.

Using these functions, we define a connection of the three Poiseuille flows as follows.

$$W(x_1, x_2) = \begin{cases} v_0 & \text{in} \quad \Omega_0 \\ \mu U_i & \text{in} \quad \Omega_i \ \ (i = 1, 2) \\ \lambda U & \text{in} \quad \Omega_3. \end{cases} \tag{0.4}$$

This W belongs to $\mathcal{P}$. Furthermore there exists a stream function $\Phi(x)$ of W

$$W = \nabla^\perp \Phi.$$

We can choose $\Phi(x)$ such that $\Phi(x)|_{\Omega_0}$ belongs to $H^2(\Omega_0)$, and is continuous and bounded in $\overline{\Omega_i}$ for $i = 1, 2, 3$.

Let W be the symmetric flow defined in (0.4). This function W does not belong to $V^S(\Omega)$, but we can construct a function $s_0 \in C_{0,\sigma}^{\infty,S}(\Omega)$ enjoying the following estimate(Lemma 3). Let $\theta(t)$ be a smooth function of $t \geq 0$ such that

$$0 \leq \theta(t) \leq 1, \quad \theta(t) = 0 \ (0 \leq t \leq 1/2), \quad \theta(t) = 1 \ (1 \leq t)$$

and we define

$$\theta_{1,\delta}(x) = \theta_\delta(x_1 \sin\alpha + x_2 \cos\alpha)$$

$$\theta_{2,\delta}(x) = \theta_\delta(-x_1 \sin\alpha + x_2 \cos\alpha)$$

$$\theta_{3,\delta}(x) = \theta_\delta(-x_2)$$

where $\theta_\delta(t) = \theta(\delta t)$.

Lemma 3. *There exists a positive constant C_0 such that for every positive ε and $0 < \delta < (2R_0)^{-1}$, there exists a function $s_0 \in C_{0,\sigma}^{\infty,S}(\Omega)$ satisfying the following estimates.*

$$((w\cdot\nabla)w, W)_\Omega \le ((w\cdot\nabla)w, s_0)_\Omega + \sum_{i=1}^{3}((w\cdot\nabla)w, \theta_{i,\delta}^2 W)_{\Omega_i} \tag{0.5}$$

$$+(\varepsilon + C_0\delta)||\nabla w||^2 \quad (\forall w \in V(\Omega)).$$

The proof is similar to that of Lemma 6 in [4] and is omitted.

In order to prove Theorem 1, we need the next theorem due to Amick[1]. Let us recall that σ and σ^S are defined by (0.2), (0.3).

Theorem 3. $\displaystyle\lim_{\delta\to+0}\Gamma(\delta) = \max\{|\mu|\sigma, |\lambda|\sigma^S\}$
where

$$\Gamma(\delta) := \sup_{w\in V^S(\Omega)} \frac{\sum_{i=1}^{3}((w\cdot\nabla)w, \theta_{i,\delta}^2 W)_{\Omega_i}}{||\nabla w||_\Omega^2}. \tag{0.6}$$

4 Proof of Theorem 1

By the assumption $\nu - \max(|\mu|\sigma, |\lambda|\sigma^S) > 0$ and Theorem 3, we can choose ε and δ such that the following inequalities hold. C_0 is the constant in Lemma 3.

$$\Gamma(\delta) - \max(|\mu|\sigma, |\lambda|\sigma^S) \le \frac{1}{4}\{\nu - \max(|\mu|\sigma, |\lambda|\sigma^S)\}, \tag{0.7}$$

$$2\varepsilon + C_0\delta \le \frac{1}{4}\{\nu - \max(|\mu|\sigma, |\lambda|\sigma^S)\}. \tag{0.8}$$

Let $\Omega^n, n = 1, 2, \cdots$, be a sequence of bounded symmetric domains with smooth boundary such that $\Omega^n \subset \Omega^{n+1} \subset \Omega$, $\partial\Omega^n \cap \partial\Omega \subset \partial\Omega^{n+1} \cap \partial\Omega$, and $\Omega^n \to \Omega$ as $n \to \infty$. We suppose that the domain Ω^1 contains $\Omega \cap \{|x| < 1/\delta\}$. We consider the stationary Navier-Stokes equations in Ω^n.

$$(NS)_n \left\{ \begin{array}{rcll} (u\cdot\nabla)u &=& \nu\Delta u - \nabla p & \text{in} \quad \Omega^n, \\ \operatorname{div} u &=& 0 & \text{in} \quad \Omega^n, \end{array} \right.$$

with the boundary condition

$$(BC)_n \left\{ \begin{array}{rcll} u &=& \beta & \text{on} \quad \partial\Omega \cap \partial\Omega^n, \\ u &=& W & \text{on} \quad \partial\Omega^n \setminus \partial\Omega \end{array} \right.$$

where W is chosen as in (0.4). A function u is called a weak solution to $(NS)_n$ $(BC)_n$, if $u \in H^1(\Omega^n)$, $\operatorname{div} u = 0$,

$$\nu(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) = 0 \quad (\forall v \in V(\Omega^n)),$$

and u satisfies the boundary condition $(BC)_n$ in the trace sense.

Let u_n be a symmetric weak solution to $(NS)_n$ $(BC)_n$, the existence of which was established by Fujita[2]. Put $u_n = w_n + W + b$, where b is the solenoidal symmetric extension (in Ω) of $\beta - W$ constructed in Lemma 2. In fact, since $\beta - W$ satisfies the hypothesis of Lemma 2, there exists its solenoidal extension b for which the following inequality holds.

$$|((v \cdot \nabla)v, b)_\Omega| \le \varepsilon \|\nabla v\|_\Omega^2 \quad (\forall v \in V^S(\Omega)).$$

It should be noted that the support of this extension is compact. Without loss of generality, we can suppose that the support is contained in $\overline{\Omega_0}$. Since $\operatorname{div} w_n = 0$ and $w_n|_{\partial\Omega^n} = 0$, w_n belongs to $V^S(\Omega^n)$ and satisfies the following equation.

$$\begin{aligned} \nu(\nabla w_n, \nabla v) &+ ((w_n \cdot \nabla)w_n, v) + ((w_n \cdot \nabla)W + (W \cdot \nabla)w_n, v) \\ &+ ((w_n \cdot \nabla)b + (b \cdot \nabla)w_n, v) \\ &= (F, v) - \nu(\nabla(b + W), \nabla v) \quad (\forall v \in V(\Omega^n)), \end{aligned} \tag{0.9}$$

where $F = -\{(b \cdot \nabla)b + (W \cdot \nabla)b + (b \cdot \nabla)W + (W \cdot \nabla)W\}$.

Using a similar argument in Morimoto-Fujita[4], we can obtain an apriori estimate for w_n independent of n, that is, there exists a positive constant M independent of n for which the following inequality holds.

$$\|\nabla w_n\| \le M.$$

By the standard argument, we can show the existence of $u \in S_\sigma^{\beta,\mathcal{P}}$ which is symmetric and satisfies (0.1), and Theorem 1 is proved. C.f. Morimoto-Fujita[4]. Q.E.D.

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Regularity of Solutions to the Stokes Equations under a Certain Nonlinear Boundary Condition

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Abstract

The regularity of a solution to the variational inequality for the Stokes equation is considered. The inequality describes the steady motion of the viscous incompressible fluid under a certain unilateral constrain of friction type. Firstly the solution is approximated by solutions to a regularized problem which is introduced by Yosida's regularization for a multi-valued operator. Then we establish a regularity result to the regularized problem. The regularity of the solution to the original inequality follows by the limiting argument.

1 Introduction

The present paper is concerned with the regularity of a solution to the following problem: Find $u \in K^1_\sigma(\Omega)$ and $p \in L^2(\Omega)$ satisfying

$$a(u, v-u) - (p, \mathrm{div}\,(v-u)) \\ + j(v) - j(u) \geq (f, v-u), \quad (\forall v \in K^1(\Omega)). \quad (1.1)$$

Here and hereafter the following notation is employed: Ω is a bounded domain in $\mathbb{R}^m$, $m = 2$ or 3. The boundary $\partial\Omega$ is composed of two connected components Γ_0 and Γ. For the sake of simplicity, we assume that Γ_0 and Γ of class C^2 and that $\overline{\Gamma}_0 \cap \overline{\Gamma} = \emptyset$. The additional smoothness assumption on Γ will be specified later. We introduce

$$K^1(\Omega) = \{v \in H^1(\Omega)^m \mid v = 0 \text{ on } \Gamma_0\},$$

then $K^1_\sigma(\Omega)$ denotes the solenoidal subspace of $K^1(\Omega)$. $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^m$ according as scalar-valued functions or vector-valued functions. We set

$$a(u,v)=\frac{1}{2}\int_\Omega \sum_{1\le i,j\le m} e_{i,j}(u)e_{i,j}(v)\,dx,\quad e_{i,j}(u)=\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}$$

for $u=(u_1,\cdots,u_m)$ and $v=(v_1,\cdots,v_m)$. Finally

$$j(v)=\int_\Gamma g|v|\,ds,\quad (ds=\text{ the surface element of }\Gamma), \tag{1.2}$$

where g is a given scalar function defined on Γ.

As was described in Fujita and Kawarada [7], the variational inequality (1.1) arises in the study of the steady motions of the viscous incompressible fluid under the *frictional boundary condition*, where u denotes the flow velocity, p the pressure and f the external forces acting on the fluid, and g is called the modulus function of friction. We now review the boundary condition of this type. Let $\sigma(u,p)$ be the stress vector to Γ. That is, we let $\sigma(u,p)=S(u,p)n$, where $S(u,p)=[-p\delta_{i,j}+e_{i,j}(u)]$ stands for the stress tensor and n the unit outer normal to Γ. Then we pose on $\sigma(u,p)$ that

$$|\sigma(u,p)|\le g \tag{1.3}$$

and

$$\begin{cases} |\sigma(u,p)|<g \implies u=0, \\ |\sigma(u,p)|=g \implies \begin{cases} u=0 \text{ or } u\ne 0, \\ u\ne 0\implies \sigma(u,p)=-gu/|u| \end{cases}\end{cases} \tag{1.4}$$

almost everywhere on Γ. The classical form of the frictional boundary value problem for the Stokes equations dealt with in [7] consists of

$$-\Delta u+\nabla p=f \text{ in } \Omega,\quad \operatorname{div} u=0 \text{ in } \Omega,\quad u=0 \text{ on } \Gamma_0 \tag{1.5}$$

together with (1.3) and (1.4). The inequality (1.1) is a weak form of this problem.

Remark 1.1. To be more precise, j should be understood as the functional on $H^{1/2}(\Gamma)^m$;

$$j(\eta)=\int_\Gamma g|\eta|\,ds,\qquad (\eta\in H^{1/2}(\Gamma)^m).$$

However, for the sake of simplicity, we will regard j as the functional on $H^1(\Omega)$ through

$$j(v|_\Gamma)=\int_\Gamma g|v|_\Gamma|\,ds$$

and write it as (1.2).

The existence theorem was established in [7]. Assume that

$$f\in L^2(\Omega)^m,\quad g\in L^\infty(\Gamma),\quad g>0 \text{ a.e. } \Gamma. \tag{1.6}$$

Then (1.1) admits of a solution $\{u, p\}$. The velocity part u is unique, and the uniqueness of the pressure part p depends on cases. We shall give an example about this issue in §5.

Henceforth we write $\|\cdot\|$, $\|\cdot\|_s$ and $\|\cdot\|_{s,\Gamma}$ instead of $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Gamma)}$, respectively. The trace γv on Γ of $v \in H^1(\Omega)$ is denoted by $v|_\Gamma$, where γ stands for the trace operator from $H^1(\Omega)$ into $H^{1/2}(\Gamma)$. The meaning of $v|_{\Gamma_0}$ and $v|_{\partial\Omega}$ is similar. For vector-valued functions, as long as there is no possibility of confusion, we shall use the same symbols. C denotes various generic constant. If it depends on parameters $q_1, \cdots, q_M$ which may not be numbers, we shall indicate it by $C = C(q_1, \cdots, q_M)$. Furthermore $\partial|\cdot|$ denotes the subdifferential of the function $|z| = (z_1^2 + \cdots + z_m^2)^{1/2}$.

The main purpose of this paper is to prove the following regularity result.

Theorem 1.1. *Let Γ be of class C^3. Assume that (1.6) and moreover that $g \in H^1(\Gamma)$. Let $\{u, p\}$ be a solution of (1.1). In particular, p is any corresponding pressure of u. Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|g\|_{1,\Gamma}),$$

where $C = C(\Omega)$. Moreover we have $\sigma(u,p) \in H^{1/2}(\Gamma)^m$ and

$$-\sigma(u,p) \in g\partial|u| \quad \text{a.e. } \Gamma.$$

In order to prove Theorem 1.1, we follow the method of Brézis [2]. Namely, we approximate a solution $\{u, p\}$ of the *inequality* (1.1) by solutions $\{u_\varepsilon, p_\varepsilon\}$ of *equations* which are obtained by replacing j by a regular functional j_ε in (1.1). Then the regularity of $\{u_\varepsilon, p_\varepsilon\}$ is studied. Actually, for $\varepsilon > 0$, we introduce

$$j_\varepsilon(v) = \int_\Gamma g\rho_\varepsilon(v)\, ds, \qquad (v \in H^1(\Omega)^m),$$

where

$$\rho_\varepsilon(v) = \begin{cases} |v| - \varepsilon/2 & (|v| > \varepsilon), \\ |v|^2/(2\varepsilon) & (|v| \leq \varepsilon). \end{cases}$$

Then an approximate problem for (1.1) we shall consider is: Find $u_\varepsilon \in K^1_\sigma(\Omega)$ and $p_\varepsilon \in L^2(\Omega)$ satisfying

$$\begin{aligned} a(u_\varepsilon, v - u_\varepsilon) &- (p_\varepsilon, \operatorname{div}(v - u_\varepsilon)) \\ &+ j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) \geq (f, v - u_\varepsilon), \quad (\forall v \in K^1(\Omega)). \end{aligned} \tag{1.7}$$

Concerning the approximate problem, we have the following three theorems.

Theorem 1.2. *Assume that (1.6) and let $\varepsilon > 0$. Then (1.7) admits a unique solution $\{u_\varepsilon, p_\varepsilon\}$ with*

$$\|u_\varepsilon\|_1 + \|p_\varepsilon\| \leq C(\|f\| + \|g\|_{L^2(\Gamma)}),$$

where $C = C(\Omega)$. *Furthermore,* $\{u_\varepsilon, p_\varepsilon\}$ *is a weak solution of (1.5) together with*

$$-\sigma(u_\varepsilon, p_\varepsilon) = g\alpha_\varepsilon(u_\varepsilon) \quad \text{a. e. } \Gamma. \quad (\textit{In particular } \sigma(u_\varepsilon, p_\varepsilon) \in L^2(\Gamma)^m), \tag{1.8}$$

where we have put

$$\alpha_\varepsilon(v) = \begin{cases} v/|v| & (|v| > \varepsilon) \\ v/\varepsilon & (|v| \le \varepsilon). \end{cases}$$

Remark 1.2. In Theorem 1.2, $\sigma(u,p)$ is understood as a functional on $H^{1/2}(\Gamma)^m$ defined by

$$\langle \sigma, \eta \rangle = a(u_\varepsilon, \psi_\eta) - (p_\varepsilon, \operatorname{div} \psi_\eta) - (f, \psi_\eta), \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi_\eta \in K^1(\Omega)$ is any extension of η.

Remark 1.3. Our choice of the regularized functional is based on *Yosida's regularization.* Namely, Yosida's regularization of $\partial|\cdot|$ coincides with $\alpha_\varepsilon = \nabla \rho_\varepsilon$.

Theorem 1.3. *Assume that (1.6) and let* $\varepsilon > 0$. *Let* $\{u, p\}$ *and* $\{u_\varepsilon, p_\varepsilon\}$ *be solutions of (1.1) and (1.7), respectively. Then we have:*

$$\|u_\varepsilon - u\|_1 + \|\tilde{p}_\varepsilon - \tilde{p}\| \le C(\Omega, g)\sqrt{\varepsilon}, \tag{1.9}$$

where $\tilde{p} = p - |\Omega|^{-1}(p, 1)$, $\tilde{p_\varepsilon} = p_\varepsilon - |\Omega|^{-1}(p_\varepsilon, 1)$ *and* $|\Omega|$ *indicates the measure of* Ω *in* $\mathbb{R}^m$

Theorem 1.4. *Let* Γ *be of class* C^3 *and let* $\varepsilon > 0$. *Assume that (1.6) and moreover that* $g \in H^1(\Gamma)$. *Let* $\{u_\varepsilon, p_\varepsilon\}$ *be a solution of (1.7). Then* $u_\varepsilon \in H^2(\Omega)^m$ *and* $p_\varepsilon \in H^1(\Omega)$ *with*

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \le C(\|f\| + \|g\|_{1,\Gamma}). \tag{1.10}$$

It should be kept in mind that $C = C(\Omega)$ *does not depend on* ε.

Now we can state:

Proof of Theorem 1.1. Let $\varepsilon > 0$, and let $\{u_\varepsilon, p_\varepsilon\}$ be a solution of (1.7). By virtue of Theorem 1.4, sequences $\|u_\varepsilon\|_2$ and $\|p_\varepsilon\|_1$ are bounded as $\varepsilon \downarrow 0$, respectively. Hence, there are subsequences $\{u_{\varepsilon'}\}$ and $\{p_{\varepsilon'}\}$ such that

$$u_{\varepsilon'} \to u^* \text{ weakly in } H^2(\Omega)^m, \quad p_{\varepsilon'} \to p^* \text{ weakly in } H^1(\Omega)$$

and

$$\|u^*\|_2 + \|p^*\|_1 \le C(\|f\| + \|g\|_{1,\Gamma}).$$

According to Theorem 1.3, $\{u^*, p^*\}$ is a solution of (1.1). Next let $\{u, p\}$ be any solution of (1.1). By the uniqueness of the velocity part, we have $u = u^*$. On the other hand, $p - p^* = k$ and a constant k is restricted via (1.3). Therefore $p \in H^1(\Omega)$, and we deduce

$$\sigma(u, p) - \sigma(u, p^*) = kn \quad \text{a. e. } \Gamma.$$

This, together with (1.3), implies that $|k| \leq 2g$ holds almost everywhere on Γ. Hence $|k| \leq 2|\Gamma|^{-1/2}\|g\|_{L^2(\Gamma)}$, where $|\Gamma|$ denotes the measure of Γ in $\mathbb{R}^{m-1}$. By making use of this estimate, we have

$$\begin{aligned} \|u\|_2 + \|p\|_1 &\leq \|u\|_2 + \|p^*\|_1 + |k|\sqrt{|\Omega|} \\ &\leq C(\|f\| + \|g\|_{1,\Gamma}), \end{aligned}$$

which completes the proof. □

The rest of the present paper is composed of the following sections:

§2. Proof of Theorem 1.2
§3. Proof of Theorem 1.3
§4. Proof of Theorem 1.4
§5. Remarks

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2 Proof of Theorem 1.2

As a preliminary to prove Theorem 1.2, we describe a decomposition theorem concerning $K^1(\Omega)$ which is essentially due to Solonnikov and Ščadilov [14].

Through Riesz's representation theorem, we define an operator B from $L^2(\Omega)$ to $K^1(\Omega)$ by

$$(Bq, v)_{H^1(\Omega)^m} = (q, \mathrm{div}\ v), \quad (\forall q \in L^2(\Omega);\ \forall v \in K^1(\Omega)). \tag{2.1}$$

Lemma 2.1. *The range $R(B)$ of B forms a closed subspace of $K^1(\Omega)$. Moreover we have the orthogonal decomposition*

$$K^1(\Omega) = R(B) \oplus K^1_\sigma(\Omega).$$

Proof. By taking $v = Bq$ in (2.1), we have $\|Bq\|_1 \leq \|q\|$; Thus, the linear operator B is bounded. On the other hand, as will be verified later, for any $q \in L^2(\Omega)$, we can take $\tilde{v} \in K^1(\Omega)$ satisfying

$$\mathrm{div}\ \tilde{v} = q \text{ in } \Omega, \quad \|\tilde{v}\|_1 \leq C'\|q\|.$$

Substituting $v = \tilde{v}$ into (2.1), we obtain $\|q\| \leq C'\|Bq\|_1$, which means that the inverse operator B^{-1} is also bounded. Consequently, we have

$$C\|q\| \leq \|Bq\|_1 \leq C'\|q\| \quad (\forall q \in L^2(\Omega)). \tag{2.2}$$

Therefore the closedness of $R(B)$ follows. To prove $R(B)^\perp = K^1_\sigma(\Omega)$ is an easy task, where $R(B)^\perp$ denotes the orthogonal complement of $R(B)$ in $K^1(\Omega)$. It remains to

verify the existence of $\tilde{v}$ appeared above. We take $w \in H^2(\Omega)$ subject to $\Delta w = p$ in Ω and introduce $\zeta \in H^{1/2}(\partial\Omega)^m$ by

$$\zeta = \begin{cases} \nabla w & \text{on } \Gamma_0 \\ \lambda n & \text{on } \Gamma, \end{cases}$$

where a constant λ is chosen as $\lambda = |\Gamma|^{-1}(\nabla w, n)_{L^2(\Gamma_0)}$. Then, since $(\zeta, n)_{L^2(\partial\Omega)} = 0$, there is a $v_0 \in H^1(\Omega)^m$ such that div $v_0 = 0$ in Ω and $\|v_0\|_1 \leq C\|\zeta\|_{1/2,\partial\Omega} \leq C\|p\|$. The desired function is $\tilde{v} = \nabla w - v_0$. □

Proof of Theorem 1.2. Since j_ε is a convex, lower semi-continuous proper functional, from standard theory of convex analysis (for example, Glowinski [8]), the minimization problem: Find $u \in K^1_\sigma(\Omega)$ satisfying

$$\mathcal{J}_\varepsilon(u) = \inf_{v \in K^1_\sigma(\Omega)} \mathcal{J}_\varepsilon(v), \quad \mathcal{J}_\varepsilon(v) = \frac{1}{2}a(v,v) - (f,v) + j_\varepsilon(v)$$

has a unique solution u which is characterized by

$$a(u, v-u) + j_\varepsilon(v) - j_\varepsilon(u) \geq (f, v-u), \quad (\forall v \in K^1_\sigma(\Omega)). \tag{2.3}$$

Following Solonnikov and Ščadilov [14], we shall prove that a scalar function p can be taken as $\{u,p\}$ solves (1.7). Substituting into (2.3) $v = u \pm t\phi$ with arbitrary $\phi \in K^1_\sigma(\Omega)$, $t > 0$ and letting $t \to 0$, we obtain

$$a(u,\phi) + \int_\Gamma g\alpha_\varepsilon(u)\cdot\phi\, ds = (f,\phi), \quad (\forall \phi \in K^1_\sigma(\Omega)).$$

We introduce a linear functional in $K^1(\Omega)$ by setting

$$F(\psi) = a(u,\psi) + \int_\Gamma g\alpha_\varepsilon(u)\cdot\psi\, ds - (f,\psi) \quad (\forall \psi \in K^1(\Omega)).$$

Then by making use of Lemma 2.1 and Riesz's representation theorem, we can easily show that there exists a $p \in L^2(\Omega)$ satisfying

$$F(\psi) = (p, \operatorname{div} \psi) \quad (\forall \psi \in K^1(\Omega)).$$

This yields

$$a(u,\psi) - (p, \operatorname{div}\psi) + \int_\Gamma g\alpha_\varepsilon(u)\cdot\psi\, ds = (f,\psi) \quad (\forall \psi \in K^1(\Omega)). \tag{2.4}$$

The uniqueness of p is obvious on account of (2.2). Thanks to the convexity of j_ε,

$$\int_\Gamma g\alpha_\varepsilon(v)\cdot(w-v)\, ds \leq j_\varepsilon(w) - j_\varepsilon(v), \quad (\forall v, w \in H^1(\Omega)^m).$$

By using this, we can easy verify that $\{u,p\}$ solves (1.7). On the other hand, (2.4) leads to

$$\langle \sigma, \psi \rangle + \int_\Gamma g\alpha_\varepsilon(u) \cdot \psi \, ds = 0 \quad (\forall \eta \in H^{1/2}(\Gamma)^m),$$

where $\psi \in K^1(\Omega)$ is any extension of η. Consequently, it follows from $g\alpha_\varepsilon(u) \in L^2(\Gamma)^m$ that $\sigma(u,p) \in L^2(\Gamma)^m$ and

$$-\sigma(u,p) = g\alpha_\varepsilon(u) \quad \text{a. e. } \Gamma,$$

which completes the proof. □

Remark 2.1. The bilinear form a is continuous in $H^1(\Omega)$; There is a constant $\delta_0 > 0$ depending only on Ω such that

$$a(u,v) \leq \delta_0 \|u\|_1 \|v\|_1 \quad (\forall u, v \in H^1(\Omega)^m).$$

Moreover, a is coercive in $K^1(\Omega)$. In fact, Korn's inequality (e.g., for example, Duvaut and Lions [5]) implies that

$$a(v,v) \geq \delta_1 \|v\|_1^2 \quad (\forall v \in K^1(\Omega))$$

with a domain constant $\delta_1 > 0$.

3 Proof of Theorem 1.3

The derivation of

$$\|u - u_\varepsilon\|_1 \leq C(\Omega, g)\sqrt{\varepsilon} \tag{3.1}$$

is same as the proof of Theorem 10.4 in Kikuchi and Oden [9]. In particular, we can take a constant as $C(\Omega, g) = (|\Gamma| \cdot \|g\|_{L^\infty(\Gamma)}/\delta_1)^{1/2}$. We proceed to the estimate of the pressure part; Namely, we shall prove

$$\|\tilde{p}_\varepsilon - \tilde{p}\| \leq C'(\Omega, g)\sqrt{\varepsilon}. \tag{3.2}$$

Putting $q_\varepsilon = \tilde{p}_\varepsilon - \tilde{p}$, we have

$$a(u - u_\varepsilon, \phi) = (q_\varepsilon, \text{div } \phi) \quad (\forall \phi \in H_0^1(\Omega)^m). \tag{3.3}$$

In view of Babušca-Aziz's lemma ([1]), we can take $w_\varepsilon \in H_0^1(\Omega)^m$ subject to div $w_\varepsilon = q_\varepsilon$ in Ω and $\|w_\varepsilon\|_1 \leq C''(\Omega)\|q_\varepsilon\|$. Now substituting $\phi = w_\varepsilon$ into (3.3), we deduce

$$\|q_\varepsilon\|^2 = a(u - u_\varepsilon, w_\varepsilon) \leq \delta_0 \|u - u_\varepsilon\|_1 \|w_\varepsilon\|_1 \leq \delta_0 C''(\Omega) \|u - u_\varepsilon\|_1 \|q_\varepsilon\|.$$

Combining this with (3.1), we get (3.2) and therefore (1.9). □

4 Proof of Theorem 1.4

We firstly review the well-known regularity result for the Stokes equations under the Neumann boundary condition.

Lemma 4.1. *Let $f \in L^2(\Omega)^m$ and $\omega \in H^{1/2}(\Gamma)^m$. Suppose that $\{u,p\} \in H^1(\Omega)^m \times L^2(\Omega)$ is a weak solution of (1.5) with $\sigma(u,p) = \omega$ on Γ. (See Remark 1.2). Then $u \in H^2(\Omega)^m$ and $p \in H^1(\Omega)$ with*

$$\|u\|_2 + \|p\|_1 \leq C(\|f\| + \|\omega\|_{1/2,\Gamma}).$$

Lemma 4.1 in the case of $\omega \equiv 0$ was described in Solonnikov [13] with a mention on Solonnikov and Ščadilov [14] concerning the method of the proof. However it seems that the complete proof for the case of $\omega \not\equiv 0$ is not explicitly stated in these papers; In this connection, we refer to a forthcoming paper Saito [11].

Lemma 4.2. *Let $\{u,p\}$ be a solution of (1.7), and put $\omega_\varepsilon = g\alpha_\varepsilon(u_\varepsilon)$. Then we have $\omega_\varepsilon \in H^{1/2}(\Gamma)^m$. Thus, from Lemma 4.1, $u_\varepsilon \in H^2(\Omega)^m$ and $p_\varepsilon \in H^1(\Omega)$.*

Proof. Firstly we verify that $\alpha_\varepsilon(u_\varepsilon) \in H^{1/2}(\Gamma)^m$ with

$$\|\alpha_\varepsilon(u_\varepsilon)\|_{1/2,\Gamma} \leq C(\varepsilon,\Gamma)\|u_\varepsilon|_\Gamma\|_{1/2,\Gamma}. \tag{4.1}$$

This is essentially due to Brézis [2], where he dealt with the scalar case, and it is possible to extend his result into our vector-values case. In fact, we can prove that

1° α_ε is a bounded operator in $H^1(\Gamma)^m$.

2° $\|\alpha_\varepsilon(\omega)\|_{1,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{1,\Gamma}$, $(\forall\omega \in H^1(\Gamma)^m)$.

3° $\|\alpha_\varepsilon(\omega) - \alpha_\varepsilon(\tilde{\omega})\|_{L^2(\Gamma)^m} \leq \varepsilon^{-1}C(\Gamma)\|\omega - \tilde{\omega}\|_{L^2(\Gamma)^m}$, $(\forall\omega,\tilde{\omega} \in L^2(\Gamma)^m)$.

The assertions 2° and 3° are easy consequences of a property of Yosida's regularization. As a result, we can apply a nonlinear interpolation theorem by J.L. Lions (Theorem 3.1, [10]) and obtain that α_ε is an operator on $H^\theta(\Gamma)^m$ with

$$\|\alpha_\varepsilon(\omega)\|_{\theta,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{\theta,\Gamma}, \quad (\forall\omega \in H^\theta(\Gamma)^m)$$

where $0 \leq \theta \leq 1$. Taking $\theta = 1/2$, we have

$$\|\alpha_\varepsilon(\omega)\|_{1/2,\Gamma} \leq \varepsilon^{-1}C(\Gamma)\|\omega\|_{1/2,\Gamma}, \quad (\forall\omega \in H^{1/2}(\Gamma)^m),$$

which implies (4.1).

Next let us denote by $\tilde{g} \in H^1(\Omega)$ the weak harmonic extension of $g \in H^{1/2}(\Gamma)$:

$$\Delta\tilde{g} = 0 \text{ in } \Omega, \quad \tilde{g} = 0 \text{ on } \Gamma_0, \quad \tilde{g} = g \text{ on } \Gamma.$$

It follows from the maximum principle that $\|\tilde{g}\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\Gamma)}$. On the other hand, we take the weak harmonic extension $\tilde{\alpha}_\varepsilon \in H^1(\Omega)$ of $\alpha_\varepsilon(v)$. That is, we extend each component of $\alpha_\varepsilon(v)$ into Ω by the harmonic function. By the definition of α_ε and again the maximum principle, we have $\|\tilde{\alpha}_\varepsilon\|_{L^\infty(\Omega)} \leq \|\alpha_\varepsilon(v)\|_{L^\infty(\Gamma)} \leq m$. Since $\tilde{g}\tilde{\alpha}_\varepsilon \in H^1(\Omega)^m$, the trace $g\alpha_\varepsilon(v) \in H^{1/2}(\Gamma)^m$. □

In order to derive the estimate (1.10) which is independent of ε, we need another device.

Lemma 4.3. *Under the same assumptions of Theorem 1.4,*

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \leq C(\Omega)(\|f\| + \|g\|_{1,\Gamma}), \tag{4.2}$$

where $\beta_\varepsilon = u_\varepsilon|_\Gamma$.

At this stage, we can state

Proof of Theorem 1.4. Let $\varepsilon > 0$. Lemma 4.3 implies that a solution $\{u_\varepsilon, p_\varepsilon\}$ of (1.7) satisfies the usual Dirichlet boundary value problem composed of (1.5) and

$$u_\varepsilon = \beta_\varepsilon \quad \text{on } \Gamma.$$

Therefore, by virtue of Cattabriga's regularity result ([3]), we have

$$\|u_\varepsilon\|_2 + \|p_\varepsilon\|_1 \leq C(\Omega)(\|f\| + \|\beta_\varepsilon\|_{3/2,\Gamma}).$$

By taking account of (4.2), we finally have (1.10). □

It remains to prove Lemma 4.3.

Proof of Lemma 4.3. Let $x_0 \in \Gamma$. Then there exist a neighborhood $U \subset \mathbb{R}^m_x$ of x_0 and a one-to-one mapping $\Phi = (\Phi_1, \cdots, \Phi_m)$ from U onto $\tilde{U} \subset \mathbb{R}^m_y$ enjoying the following properties (See, for example, §I-2 in Wolka [15]):

1° Φ is a C^3-diffeomorphism;

2° $\Phi(x_0) = 0$;

3° $\Phi(U \cap \Omega) = Q_R \equiv \{y = (y', y_m) \in \mathbb{R}^{m-1} \times \mathbb{R};\ |y'| < R,\ 0 < y_m < R\}$;

4° $\Phi(U \cap \Gamma) = \gamma_R \equiv \{y = (y', y_m) \in \mathbb{R}^{m-1} \times \mathbb{R};\ |y'| < R,\ y_m = 0\}$;

5° $\dfrac{\partial \Phi_m}{\partial x_j} = -n_j$ on $U \cap \Gamma$, where $n = (n_1, \cdots, n_m)$.

Setting $y = \Phi(x) = (\Phi_1(x) \cdots, \Phi_m(x))$ and introducing

$$v(y) = u(x), \qquad q(y) = p(x)$$

and

$$\tilde{f}(y) = f(x), \quad \tilde{g}(y) = g(x), \quad \tilde{\alpha}(v(y)) = \alpha(u(x)),$$

we have for $j = 1, \cdots, m$

$$-\sum_{1\leq k,l\leq m} \frac{\partial}{\partial y_l} E_{j,k,l} + \sum_{k=1}^{m} \frac{\partial \Phi_k}{\partial x_j}\frac{\partial q}{\partial y_k} = F_j \qquad \text{in } Q_R, \tag{4.3}$$

$$\tilde{\sigma}_j(v, q) = -\tilde{g}\tilde{\alpha}_j(v) \qquad \text{on } \gamma_R \tag{4.4}$$

and

$$\sum_{1\le k,l\le m} \frac{\partial \Phi_k}{\partial x_l}\frac{\partial v_l}{\partial y_k} = 0 \text{ in } Q_R. \tag{4.5}$$

Here

$$E_{j,k,l} = b_{kl}\frac{\partial v_j}{\partial y_k} + \frac{\partial \Phi_k}{\partial x_j}\sum_{\nu=1}^m \frac{\partial \Phi_l}{\partial x_\nu}\frac{\partial v_\nu}{\partial y_k}, \qquad b_{kl} = \sum_{\nu=1}^m \frac{\partial \Phi_k}{\partial x_\nu}\frac{\partial \Phi_l}{\partial x_\nu},$$

$$F_j = \tilde{f}_j - \sum_{\nu=1}^m d_\nu \frac{\partial v_j}{\partial y_\nu}, \qquad \tilde{\sigma}_j(v,q) = q\frac{\partial \Phi_m}{\partial x_j} - \sum_{\nu=1}^m E_{j,\nu,m}$$

and d_j denotes a bounded function in Q_R depending on Φ, $\nabla\Phi$ and $\nabla^2\Phi$. In fact, (4.3) follows from

$$-\sum_{l=1}^m \frac{\partial}{\partial x_l} e_{j,l}(u) + \frac{\partial p}{\partial x_j} = f_j,$$

which is an equivalent expression to $-\Delta u_j + \partial p/\partial x_j = f_j$ under div $u = 0$. On the other hand, div $u = 0$ and (1.8) imply (4.5) and (4.4), respectively.

Let φ_j be a function such that $\varphi_j(y', y_m) = 0$ for $|y'| \ge R$ or for $y_m \ge R$. Multiplying by φ_j both sides of (4.3) and integrating over Q_R, we have by integration by parts

$$\sum_{k,l}\int_{Q_R} E_{j,k,l}\frac{\partial \varphi_j}{\partial y_l} - \sum_k \int_{Q_R} q\frac{\partial}{\partial y_k}\Big(\frac{\partial \Phi_k}{\partial x_j}\varphi_j\Big) = \int_{\gamma_R} \tilde{\sigma}_j(v,q)\varphi_j\, dy' + \int_{Q_R} F_j\varphi_j.$$

Hence, writing $\varphi = (\varphi_1, \cdots, \varphi_m)$, we obtain

$$\sum_{k,l}\int_{Q_R}\Big[b_{k,l}\frac{\partial v}{\partial y_k}\frac{\partial \varphi}{\partial y_l} + \Big(\frac{\partial v}{\partial y_k}\nabla_x\Phi_l\Big)\Big(\frac{\partial \varphi}{\partial y_l}\nabla_x\Phi_k\Big)\Big]$$

$$-\sum_k \int_{Q_R} q\frac{\partial}{\partial y_k}\Big(\varphi\nabla_x\Phi_k\Big) = \int_{\gamma_R}\tilde{\sigma}(v,q)\varphi\, dy' + \int_{Q_R} F\varphi. \tag{4.6}$$

Now we choose φ_j as

$$\varphi_j = -\sum_{i=1}^{m-1}\frac{\partial}{\partial y_i}\Big(\eta^2\frac{\partial v_j}{\partial y_i}\Big),$$

where $\eta \in C^\infty(\mathbb{R}^m_y)$ stands for a cut-off function subject to

$$0 \le \eta \le 1 \text{ in } \mathbb{R}^m_y, \quad \eta = 1 \text{ in } Q_{R/2}, \quad \eta = 0 \text{ for } |y'| \ge R \text{ or } y_m \ge R.$$

If we write (4.6) as $I_1 - I_2 = I_3 + I_4$, then we deduce

$$\begin{aligned} I_1 \geq c_1 \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\|^2 \\ - c_2 \|v\|_{1,Q_R} \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\| - c_3 \|v\|_{1,Q_R}^2, \end{aligned} \tag{4.7}$$

$$|I_2| \leq c_4 \|q\|_{Q_R} \Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\| \Big), \tag{4.8}$$

$$I_3 \leq c_5 \|\tilde{g}\|_{1,\gamma_R} \|v\|_{1,Q_R}, \tag{4.9}$$

and

$$I_4 \leq c_6 (\|\tilde{f}\|_{Q_R} + \|v\|_{1,Q_R}) \Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\| \Big) \tag{4.10}$$

Here we check only (4.9). Other inequalities are verified by Saito [12]. Firstly, we note that

$$\nabla_s \tilde{\alpha}(s) t \cdot t \geq 0 \qquad \forall s, t \in \mathbb{R}^m.$$

Hence

$$\begin{aligned} I_3 &= -\sum_{i=1}^{m-1} \int_{\gamma_R} \left\{ \tilde{g} \eta^2 \Big[(\nabla_v \tilde{\alpha}(v)) \frac{\partial v}{\partial y_i} \cdot \frac{\partial v}{\partial y_i} \Big] + \Big(\frac{\partial \tilde{g}}{\partial y_i} \tilde{\alpha}(v) \Big) \cdot \eta^2 \frac{\partial v}{\partial y_i} \right\} dy' \\ &\leq \sum_{i=1}^{m-1} \left\| \frac{\partial \tilde{g}}{\partial y_i} \right\|_{\gamma_R} \left\| \frac{\partial v}{\partial y_i} \right\|_{\gamma_R} \leq c_7 \|\tilde{g}\|_{1,\gamma_R} \|\nabla v\|_{Q_R}. \end{aligned}$$

Taking into account of (4.7)-(4.10), we have

$$\begin{aligned} \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\|^2 &\leq c_8 (\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R}) \cdot \\ &\qquad \Big(\|v\|_{1,Q_R} + \sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\| \Big). \end{aligned}$$

This yields

$$\sum_{i=1}^{m-1} \sum_{k=1}^{m} \left\| \eta \frac{\partial^2 v}{\partial y_i \partial y_k} \right\| \leq c_9 (\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R}),$$

and therefore v

$$\sum_{i=1}^{m-1} \left\| \frac{\partial v}{\partial y_i} \right\|_{1,Q_{R'}} \leq c_{10} (\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R})$$

with $R' = R/2$. According to this inequality, introducing

$$\mu_j = \frac{\partial v}{\partial y_j}\Big|_{\gamma_{R'}}, \quad (j = 1, \cdots, m-1)$$

we arrive at

$$\|\mu_j\|_{1/2,\gamma_{R'}} \le c_{11}(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R}), \quad (j = 1, \cdots, m-1).$$

This means that all tangential derivatives of $v|_{\gamma_{R'}}$ belongs to $H^{1/2}(\gamma_{R'})$. Therefore, $v|_{\gamma_{R'}} \in H^{3/2}(\gamma_{R'})^m$ and

$$\|v|_{\gamma_{R'}}\|_{3/2,\gamma_{R'}} \le c_{12}(\|v\|_{1,Q_R} + \|q\|_{Q_R} + \|\tilde{g}\|_{1,\gamma_R} + \|\tilde{f}\|_{Q_R})$$

Summing up the above estimates, we finally have

$$\|\beta_\varepsilon\|_{3/2,\Gamma} \le c_{13}(\|u\|_{1,\Omega} + \|p\|_{\Omega} + \|g\|_{1,\Gamma} + \|f\|_{\Omega}).$$

Therefore we have established the desired (4.2). □

Remark 4.1. As is described in [12], the inequality (4.7) is a consequence of Korn's inequality, and (4.8) is that of (4.5). Furthermore, the constant c_4 depends on the third derivatives of Φ so that we need to assume that Γ is of class C^3.

5 Remarks

(A) Non-uniqueness of the pressure part. In general, a corresponding pressure p of the velocity u which is a solution of (1.5)(1.3)(1.4) is not unique. We give a simple example. For the time being, we employ the polar coordinates $x = (r, \theta)$ in $\mathbb{R}^2$. We assume that

$$\Omega = \{(r,\theta); 1 < r < \sqrt{2}\}, \quad \Gamma_0 = \{r = 1\}, \quad \Gamma = \{r = \sqrt{2}\},$$

and set $e_r = (\cos\theta, \sin\theta)$, $e_\theta = (-\sin\theta, \cos\theta)$. Put $u(r,\theta) = w(r)e_\theta$ and $p(r) = \kappa r$, where $w(r) = 2/r - r$ and $\kappa > 0$ is a constant. We notice that $\{u, p\}$ solves

$$-\Delta u + \nabla p = \kappa e_r, \quad \text{div } u = 0 \text{ in } \Omega, \quad u|_{\Gamma_0} = e_\theta, \quad u|_\Gamma = 0. \tag{5.1}$$

Moreover the stress vector is written as $\sigma(u,p) = -p(r)e_r + W(r)e_\theta$, where $W(r) = -4w(r)/r^2$. Hence $|\sigma(u,p)| = \sqrt{2\kappa^2 + 4}$.

Firstly we define $g = \sqrt{2\kappa^2 + 4} + 1$. Then $|\sigma(u,p)| < g$ and $u|_\Gamma = 0$ hold. Therefore $\{u,p\}$ is a solution of (5.1)(1.3)(1.4).

Next put $p_c = p + c$ with a constant c, and let $c_0 < c_1$ be roots of the equation $2\sqrt{2}\kappa c + c^2 = 1$. Then the following facts are hold true: (i) if $c_0 \le c \le c_1$ then $\{u, p_c\}$ is also a solution of (5.1)(1.3)(1.4). (ii) if $c < c_0$ or $c_1 < c$ then $\{u, p_c\}$ is not a solution. Namely, in this case, p is not unique and the non-uniqueness is restricted through (1.3).

On the other hand, if non-trivial movement ($u \ne 0$) takes place on a portion $\Gamma_1 \subset \Gamma$, then p is uniquely determined.

(B) Other flow problems. From the view point of physics, some modifications of (1.1) are much more interesting. Actually, H. Fujita [6] studied the Navier-Stokes or the Stokes equations under the *leak or slip boundary conditions of friction type.* The similar results about the regularities of solutions for these problems could be obtained by the same method presented in this paper. See, for more detail, Saito [12].

(C) Problems in elasticity theory. Our method is applicable for some problems in elasticity theory. For instance, we can give another proof of a regularity theorem described in M. Cocu and A.R. Radoslovescu [4], where they dealt with Signorini problem with non-local friction.

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Viscous incompressible flow in unbounded domains

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Abstract

We prove the existence of a weak solution for a two-phase problem for viscous incompressible fluids in domains with noncompact boundary.

1 Introduction

Flow problems with moving boundary are very interesting phenomena which are often found in the nature and engineering applications. These problems consist of the free surface flows ([8]), flows with interface ([3],[4],[9]) and flows in a moving domain with an assigned law ([5],[6],[7]). In this paper we consider the solvability of the two-phase problem or flow with free interface for viscous incompressible fluids in domains with noncompact boundaries. Flow problems in unbounded domains present interesting peculiar behaviour such that a particular attention was given to such a problem recently (see [1] for references). In fact posing boundary or initial-boundary value problems for the Navier-Stokes equations, it is necessary to prescribe, besides the usual boundary conditions, certain functionals of the solution. This condition is connected with the possible non coincidence of certain solenoidal spaces ([1]). Due to the existence theory, a particular attention was given to this problem in the Sobolev spaces $W^{1,p}$.

In this paper, first we study the non coincidence of the solenoidal spaces H and $\hat{H}$ in L^2(see below for the definitions) and we characterize the subspace $\hat{H}/H$. Then we pass to consider the two-phase problem. We consider the free two-phase problem in unbounded domains with exits to infinity, prescribing additional conditions on the interface Γ.

The paper is organized as follows.

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In section 2 we give notations and preliminary results. We present the two phase flow in unbounded domains having several outlets at infinity. We assume that the two fluids are both viscous, incompressible and immiscible, and the physical effect of the surface tension at the interface is considered. Further we assume that the densities and the viscosities are constant. We reduce this problem to one-fluid flow and we state the results. In section 3 we consider the non coincidence of the solenoical spaces in L^2 and characterize the space difference. In section 4 we prove the existence of a weak solution.

2 Preliminaries

Let $D \in R^n$ with boundary ∂D . Consider a family of bounded subdomains D_τ of D depending on the parameter $\tau = (\tau_1, ..., \tau_N)$ which ranges over the parallelepiped

$$G = \{\tau : 0 \leq \tau_i \leq \bar{\tau}_i, i = 1, ..., N\}, \bar{\tau} =$$

$$(\bar{\tau}_1, ..., \bar{\tau}_N).$$

We mainly consider the domain such that outside the sphere $|x| = R_0$ it splits into m connected components D_i (exits to infinity), i.e. $D = D_0 \cup (\cup_{i=1}^m D_i)$. D is the so-called domain with m exits to infinity. Suppose that D_i, for every i, is a domain such that its intersection with the plane, orthogonal to some smooth curve L_i at the point $P(\tau_i) \in L_i$, form around that point the domain $S_{\tau_i} \equiv S_i$; the parameter τ_i being equal to the length of L_i measured from the surface $\{x : |x| = R_0\}$ to the point $P(\tau_i)$.

It is assumed that D is a smooth domain of class C^k with k a positive integer. Furthermore we assume that the unit normal vector field $n(x)$ with $x \in \partial D$ is inward pointing on ∂D. If it is necessary we consider also an extension of n in a neighborhood of $\bar{D}$. If $k \geq 2$, in each point of $x \in \partial D$, we can define the mean curvature R .

To simplify the discussion, we do not distinguish in our notations whether the functions are R-or R^m-valued and c denotes a constant. We define $C_0^\infty(D)$ to be the linear space of infinitely many times differentiable functions with compact supports in D.

We denote $\mathcal{C}_0^\infty$ the linear space of divergence free functions of C_0^∞. For any s, q, $s \geq 0, q \geq 1$, $W^{s,q}(D)$ denotes the Sobolev space of order s on $L^q(D)$. Further the norm on $W^{s,q}(D)$ is denoted by $||\phi||_{s,q}$.

When $q = 2$, $W^{s,2}(D)$ is usually denoted by $H^s(D)$ and we drop the subscript $q = 2$ when referring to its norm. $H^s(D)$ is a Hilbert space for the scalar product

$$((u, v))_s = \sum_{|\alpha| \leq s} \int_D D^\alpha u D^\alpha v dx.$$

In particular, in $L^2(D)$, we write the scalar product $(u.v)$ and the norm $|v|$.

Further we define $W_0^{s,q}(D)$ the closure of $C_0^\infty(\Omega)$ for the norm $|| \cdot ||_{s,q}$.

Let us introduce the following spaces of divergence-free functions. We denote by

$$V^s \equiv \{completion\ of\ \mathcal{C}_0^\infty\ in\ H_0^s(D)\}.$$

and it is a closed subspace of $H^s(D)$.

We set $V^1 = V$ and $V^0 = H$. Further we denote

$$\hat{V} \equiv \{v \in H_0^1(D) | \nabla \cdot v = 0\}$$

and

$$\hat{H} \equiv \{v \in L^2(D) | \nabla \cdot v = 0 \quad in \quad distribution \quad sense\}.$$

In general $V \subset \hat{V}$ and $H \subset \hat{H}$. In literature the first problem has been studied extensively (see [1] for references). The last one is not mentioned. In section 3 this problem is studied and we characterize completely the subspace

$$\frac{\hat{H}}{H.}$$

For any measurable set $\Omega \in R^3$, the characteristic function $\chi_\Omega \equiv \chi$ is defined as usual by $\chi(x) = 1$ if $x \in \Omega$ and $= 0$ if $x \in \Omega^c$ ($\Omega^c = R^3 \setminus \Omega$). If the Lebesgue measure of Ω is finite, then $\chi \in L^1(R^3)$. Let us consider the gradient $\nabla\chi$ as a distribution in R^3 defined by

$$< \nabla\chi, \phi >= - \int_\Omega \nabla \cdot \phi dx$$

for any $\phi \in C_0^\infty(R^m)$. Using the well-known Stokes' formula, we get

$$< \nabla\chi, \phi >= - \int_\Gamma \phi \cdot nd\Gamma =<^* \gamma n, \phi >$$

that is the gradient of χ takes the form

$$\nabla\chi =^* \gamma n.$$

Here γ is the trace operator : $C^k(\Omega) \to C^k(\Gamma)$ and ${}^*\gamma$ denotes its transposed operator.

In general $\nabla\chi$ is an element of $(C^k(\Omega))'$.

We remark that if $\Omega \in C^1$, then $\chi \in W^{s,q}(R^3)$ for $s < 1/q, 1 \leq q < \infty$.

In addition to the above notation we introduce the shape functional

$$J(\Omega) = meas(\Gamma) = \int_\Gamma d\Gamma.$$

If the domain $\Omega(t)$ changes in time following

$$\partial_t x = v(x,t), \; x(y,0) = y \in \Omega$$

we have

$$d_t \int_{\Gamma(t)} d\Gamma = \int_{\Gamma(t)} d\Gamma = - \int_{\Gamma(t)} v \cdot nRd\Gamma.$$

Now we consider the two-fluid flow in D. We assume that D consists of two time-dependent subdomains $\Omega^i(t)$, for $i = 1,2$ which are filled with immiscible viscous incompressible fluids (1) and (2), respectively:

$$D = \Omega^1(t) \cup \Omega^2(t)$$

for $0 \leq t < T$. $D_0 \subset \Omega^1(t)$ (bounded) and $\Omega^2(t) \subset \cup D_i$. The interface

$$\Gamma(t) = (\partial\Omega^1 \cup \partial\Omega^2) \setminus \partial D = \cup\Gamma_i(t)$$

for $0 \leq t < T$.

The flow of the two fluids is governed by the equations in $\Omega^i(t)$

$$\begin{aligned} &\rho^i(u^i_t + u^i \cdot \nabla u^i) - \mu^i \Delta u^i + \nabla p^i = \rho^i f^i, \ in\, \Omega^i(t), \ t > 0, \\ &\nabla \cdot u^i = 0, \\ &u^i(x,t) = 0, \ on\ \partial\Omega^i(t) \setminus \Gamma(t), \end{aligned} \tag{2.1}$$

with $i = 1,2$.

Here u^i is the velocity vector field of the fluid, p^i the pressure and f^i are given vector fields of the external mass forces, the viscosity coefficient μ^i and the density ρ^i are constant.

The initial conditions are

$$u^1_0 = u^1(x,0)\ in\ \Omega_1(0), \ u^2_0 = u^2(x,0) \in \ \Omega^2(0).$$

The boundary conditions on the interface $\Gamma(t)$ between the fluids (1) and (2) at the time $t > 0$ are expressed by

$$u^1 = u^2, \quad T^1 n - T^2 n = \sigma R n, \ t > 0,$$

where σ is the coefficient of the surface tension and T^i is the stress tensor with components

$$T^i_{jk} = -p^i \delta_{jk} + \mu^i(\partial_{x_k} u^i_j + \partial_{x_j} u^i_k)$$

$(j,k = 1,2,3)$, n is the unit inner normal vector to Γ pointing to $\Omega^1(t)$), and R is the mean curvature of $\Gamma(t)$.

Here and in what follows we shall use the well-known notation of vector analysis and summation convention.

In order to combine the basic equations (1.1), we introduce the characteristic functions of $\Omega^1(t), \Omega^2(t)$, namely $\chi^1 = \chi(t), \chi^2 = 1 - \chi(t)$, respectively.

The interface between two fluids is defined as the discontinuity surface of χ.

Using χ, defining $v = \chi^1 u^1 + \chi^2 u^2, \rho = \chi^1 \rho^1 + \chi^2 \rho^2, \mu = \mu^1 \chi^1 + \chi^2 \mu^1, p = \chi^1 p^1 + \chi^2 p^2$, and $f = \chi^1 \rho^1 f^1 + \chi^2 \rho^2 f^2$, and bearing in mind the boundary conditions, the Navier-Stokes system for the fluids (1),(2) are formally equivalent to the following system in D

$$\begin{aligned} &\rho(v_t + v \cdot \nabla v) - \nabla \cdot (\mu \nabla v) + \nabla p = \rho f + \sigma \nabla \chi^1 R, \\ &\nabla \cdot v = 0, v(x,t) = 0, \ x \in \partial D; \ v(x,0) = v_0, \ x \in D. \end{aligned} \tag{2.2}$$

The function $\chi \equiv \chi^1$ is considered as a scalar quantity transported by the fluid in a turbulent flow of a viscous incompressible fluid and satisfies

$$\begin{aligned} &\chi_t + v \cdot \nabla\chi = 0, \quad in \quad D, \quad t > 0 \\ &\chi(0) = \text{characteristic function of} \quad \Omega^1(0). \end{aligned} \tag{2.3}$$

For domains of type D the motion of a viscous incompressible fluid is not always uniquely determined by usual initial and boundary conditions. The necessity to prescribe additional conditions is connected with the fact that the two spaces V, $\hat{V}$ are not identical. Then we complete the kinematics conditions on $\Gamma(t)$ prescribing the net flux of the fluid on $\Gamma_i(t)$

$$\int_{\Gamma_j(t)} u^i \cdot n d\Gamma_j = \alpha^i_j(t)$$

α^i_j are given functions with the condition $\sum_{j=1}^n \alpha^i_j(t) = 0$.

For v the flux condition is

$$\int_{S_j} v \cdot n = \alpha_j(t)$$

where $S_j = D_0 \cap D_j$.

Now we give the definition of weak solutions of (1.2),(1.3). We assume that:

Assumption 1 - *There exists a divergence free vector field* $b(x,t) \in L^\infty(Q_T) \cap H^1(Q_T))$ *such that*

$$\int_{S_i} b \cdot n dS = \alpha_i(t)$$

with $b = 0$ *on* ∂D.

Here $Q_T = D \times (0,T)$. We introduce a new unknown vector field: $u = v - b$. This permits to reduce the problem (2.2),(2.3) to a similar problem with zero flux through the S_i:

$$\begin{aligned} &\rho(u_t + (u+b) \cdot \nabla u) + \rho(u+b) \cdot \nabla b \\ &-\nabla \cdot (\mu \nabla u) + \nabla p = \rho f + \sigma \nabla \chi^1 R + B, \\ &\nabla \cdot u = 0, \ u(x,t) = 0, \ x \in \partial D, \\ &v(x,0) = v_0, \ x \in D \quad u \to 0 \ as \ |x| \to \infty, \\ &\partial_t \chi + u \cdot \nabla \chi + b \cdot \nabla \chi = 0, \quad \chi(0) = \chi_0 \end{aligned} \tag{2.4}$$

where $B = -\rho \partial_t b + \nabla \cdot (\mu \nabla b)$.

Now we give the definition of weak solution of (2.4). To avoid tedious calculations we assume that $\sigma = 0$.

By a weak solution of (2.4) *with* $\sigma = 0$, *we call a couple* (u, χ) *such that, for any* $T > 0$ *and any* $\phi \in C^1(0,T; C_0^\infty(D))$, $\phi(T) = 0$, *and* $\eta \in C^1(Q_T)$, $\eta(T) = 0$,

$$\int_0^T ((\rho u, \phi_t) + (\rho(u+b)\cdot\nabla)\phi, u) - (\rho(u+b)\cdot\nabla b, \phi) -$$
$$(\mu\nabla u, \nabla\phi) + (f + B, \phi))dt + (u_0, \phi(0)) = 0,$$
$$\int_0^T (\chi, \eta_t) + (\chi, (u+b)\cdot\nabla\eta)dt + (\chi_0, \eta(0)) = 0,$$

with $\chi \in L^\infty(Q_T), \chi u \in L^\infty(0,T;L^2(D)), u \in L^2(0,T;V)$.

In section 5 we prove

Theorem 1 - *We assume that $f \in L^2(Q_T)$ and $u_0 \in H$. Then there exists a weak solution of (2.4).*

3 The spaces H and $\hat{H}$

In this section we characterize the subspace

$$\hat{H}/H.$$

We use the method to characterize the kernel of *rot* operator in non connected domain. We denote by S_i^+ and S_i^- the two sides of S_i and n_i the unit normal on S_i oriented from S_i^+ toward S_i^-; if a function v takes different values on S_i^+ and S_i^-, then we set $[v]_i = v|_{S_i^+} - v|_{S_i^-}$. Let $\tilde{D} = D \setminus \cup S_i$.

We consider the following problem:

Find function q harmonic in $\tilde{D}$ such that

$$\partial_n q = 0 \quad on \quad \partial D,$$
$$[q]_i = constant, \quad i = 1, ..., N$$
$$[\partial_{n_i} q]_i = 0, \quad i = 1, ..., N$$

and

$$q \to c \quad as \quad |x| \to \infty$$

Lemma 1 - *For $i = 1, ..., N$ there exists a function q_i unique up to an additive constant, such that*

$$\begin{aligned} &\Delta q_i = 0 \; in \; \tilde{D}, \\ &\partial_n q_i = 0 \; on \; \partial D, \\ &[\partial_n q_i]_j = 0, j = 1, ..., N, \\ &[q_i]_j = 0 \; j \neq i \\ &[q_i]_i = 1, \\ &q_i \to c_i \quad as \quad |x| \to \infty. \end{aligned} \tag{3.1}$$

The above problem is equivalent to

$$\begin{aligned}
&\Delta q_i = 0 \; in \; \tilde{D}, && (3.2)\\
&\partial_n q_i = 0 \; on \; \partial D_i, &&\\
&[\partial_n q_i]_j = 0, j = 1, ..., N, &&\\
&[q_i]_j = 0 \; j \neq i, &&\\
&[q_i]_i = (undetermined) \quad constant, && (3.3)\\
&\int_{S_i} \partial_n q_i dS = 1, &&\\
&q_i \to c_i \quad as \quad |x| \to \infty. &&
\end{aligned}$$

We notice that the above problem is variational. In fact let

$$X_i = \{\phi \in H^1(\tilde{D}), [\phi]_i = const, [\phi]_j = 0 \; \forall j \neq i\}$$

where $H^1(\tilde{D}) = \{\phi | \phi \in L^2_{loc}(\tilde{D}), \nabla\phi \in L^2(\tilde{D}\}$.

The problem consists in finding $q_i \in X_i$ such that

$$\int_D \nabla q_i \cdot \nabla \phi dx = [\phi]_i, \; \forall \phi \in X_i.$$

The left-hand side defines on X_i/R a continuous coercive bilinear form on X_i/R and the right hand side is a continuous form on X_i/R

Lemma 2 - *The space generated by $\nabla q_1, ..., \nabla q_N$ is contained in $\hat{H}$ and is orthogonal to H.*

$u_i = \nabla q_i$ (the gradient is in classical sense) is divergence free.
Let $\phi \in C_0^\infty(D)$, we have

$$(\nabla \cdot u_i, \phi) = -(\nabla q, \nabla \phi) = -\int_{D \; or \; \tilde{D}} \nabla q \nabla \phi dx =$$

$$(since \; \Delta q = 0) = \int_{\partial \tilde{D}} \partial_n q \phi d\Gamma =$$

$$\sum_1^N \int_{S_i} [\partial_n q]_i \phi dS = 0.$$

Now u_i is orthogonal to the space H.
We notice that

$$H \equiv \{v \in \hat{H}, \int_{S_i} v \cdot n dS = 0, i = 1, ..., N\}.$$

Let $v \in \hat{H}$. Then $v \in H$ if and only if $(v, \nabla q_i) = 0$, i = 1,...,N.

This is consequence of the generalized Stokes formula

$$(v, \nabla q_i) = \int_{\tilde{\Omega}} v \nabla q_i dx = \int_{\partial\tilde{\Omega}} v \cdot n q_i d\Gamma =$$

$$\int_{S_i} v \cdot n dS.$$

Now we denote H_c the space spanned by ∇q_i $i = 1, 2, ..., n$.

Lemma 3 - *H_c is the topological complement of H. The dimension of H_c is N-1 and there exists a basis $(u_1, ..., u_{N-1})$ of H_c such that*

$$\int_{S_j} u_i \cdot dS = \delta_{ij}, \quad i, j = 1, ..., (N-1),$$

$$\int_{S_N} u_i dS = -1.$$

First, $H_c \cap H = 0$. In fact if

$$\int_{S_i} u_i \cdot n_i d\Gamma = 0$$

from Lemma 1 we have $u_i = 0$.

Further H_c has finite dimension.

In fact, let $\Phi : H_c \to R^N$ the application defined by

$$\Phi(u) = (\int_{S_1} u \cdot n d\Gamma, ..., \int_{S_{N-1}} u \cdot n d\Gamma).$$

If $u \in H_c$ and $\Phi(u) = 0$, then $u \in H$ and consequently $u = 0$. Now the elements of H_c are divergence free, then

$$\sum_1^N \int_{S_i} u \cdot n d\Gamma = 0.$$

Further Φ is injective. We have that dimension of H_c is $\leq N$. But the above condition implies that u_i are linearly independent. So the dimension of H_c is $N - 1$.

It is routine matter to construct a vector $v \in H_c$ such that $v = 0$ on ∂D. First construct a solenoidal vector ϕ such that $\phi \cdot n = 0$ and the tangential component $\phi_\tau = u$ on ∂D. Then $v = u - \phi$ is in H_c with zero boundary value. v has bounded Dirichlet integral in domain with " wide " exits to infinity.

4 Proof of Theorem 1

We prove the existence of a weak solution of (2.4) using the method of the elliptic regularization (see [7]).

First we consider the following auxiliary problem:

Find $(u,\chi) \equiv (u^{m,\epsilon},\chi^{m,\epsilon})$ *such that* $\forall\phi \in H^1(Q_T)$ *with* $\nabla\cdot\phi = 0$

$$\int_0^T (\frac{1}{m}(\partial_t u, \partial_t\phi) + (\mu\nabla u, \nabla\phi) - (exp(kt)(\rho(\tilde{u}+b)\cdot\nabla\phi, u) \tag{4.1}$$
$$+(exp(kt)(\rho(\tilde{u}+b)\cdot\nabla b, \phi) + k(u,\phi) - (\rho u, \partial_t\phi))dt$$
$$+(\rho(T)u(T),\phi(T)) = \int_0^T exp(-kt)(f+B,\phi)dt + (\rho_0 u_0, \phi(0)),$$
$$\partial_t\chi + \tilde{u}\cdot\nabla\chi + b\cdot\nabla\chi = 0$$

holds.

$\tilde{u} \equiv \tilde{u}^{m,\epsilon}$ is a regularization of $u^{m,\epsilon}$ using a space-mollifier depending on ϵ.

The existence of the continuity equation is obtained as usual by the method of characteristics considering the solutions of

$$\partial_t y = (\tilde{u}+b)(y,t), \; y(t,x,t) = x.$$

To obtain the existence of u we set

$$a(u,\phi) = \int_0^T (\frac{1}{m}(\partial_t u, \partial_t\phi) + (\mu\nabla u, \nabla\phi)$$
$$-(exp(kt)(\rho(\tilde{u}+b)\cdot\nabla\phi, u) + k(u,\phi) + (exp(kt)(\rho(\tilde{u}+b)\cdot\nabla b, \phi) +$$
$$+k(u,\phi) - (\rho v, \partial_t\phi))dt + (\rho(T)u(T),\phi(T)),$$

$$L(\phi) = \int_0^T exp(-kt)(f+B,\phi)dt + (\rho_0 u_0, \phi(0)).$$

By the following well known theorem (see [2]) one obtains the existence of a solenoidal solution in $H^1(Q_T)$ of the equation

$$a(v,\phi) = L(\phi). \tag{4.2}$$

Theorem 3. *If*

i) there exists a constant $c > 0$ *such that*

$$a(u,u) \geq c\|u\|^2_{H^1(Q_T)};$$

ii) the form $u \to a(u,\phi)$ is weakly continuous in $H^1(Q_T)$ i.e. $u_n \to u$ weakly in $H^1(Q_T)$ implies

$$\lim_{n\to\infty} a(u_n,\phi) = a(u,\phi).$$

Then (4.2) has a solenoidal solution in $H^1(Q_T)$.

The condition (i) can be easily proved; in fact, bearing in mind the continuity equation,

$$a(u,u) \geq \int_0^T (1/m|\partial_t u|^2 + c\|\nabla u\|^2 + k\|u\|^2]dt + \tag{4.3}$$
$$+1/2\|\sqrt{\tilde\rho}(T)u(T)\|^2 + 1/2\|\sqrt{\tilde\rho}(0)u(0)\|^2 \geq c\|u\|_{H^1(Q_T)}.$$

For (ii) we consider $u^n \to u$ weakly in $H^1(Q_T)$. We need to prove essentially the convergence of the nonlinear term. Of course $\rho^m \to \rho$ for example in $L^2_{loc}(Q_T)$. Further $u^n \to u$ in $L^2_{loc}(Q_T)$ then

$$\rho^n\tilde u^n \cdot \nabla u^n \to \rho\tilde u \cdot \nabla u.$$

Hence

$$a(u^n,\phi) \to a(u,\phi)$$

for any $\phi \in H^1(Q_T) \cap V$.

To passing to the limit $m \to \infty, \epsilon \to 0$ in (4.1) we will need a priori estimates of the approximations $u^{m,\epsilon}$. If we replace in (4.1) ϕ by u we obtain

$$\|\chi\|_{L^\infty(Q_T)} \leq c, \quad \frac{1}{m}\int_0^T |\partial_t u^{m,\epsilon}\|_{L^2(D)}dt \leq c, \tag{4.4}$$
$$\int_0^T \|u^{m,\epsilon}\|_{H^1(D)}dt \leq c, \ \|u^{m,\epsilon}(T)\|_{L^2(D)} \leq c, \ \|u^{m,\epsilon}(0)\|_{L^2(D)} \leq c.$$

The constant in (4.4) is independent of m and ϵ.

The estimates (4.4) permit to deduce that there exists a subsequence, still denoted by $\{(\chi^{m,\epsilon}, u^{m,\epsilon})\}$ such that for $m \to \infty$

$u^{m,\epsilon} \to u^\epsilon$ *weakly in* $L^2(0,T;V)$;
$\chi^{m,\epsilon} \to \chi^\epsilon$ *weak* in* $L^\infty(Q_T)$ *and strongly in* $L^2(Q_T)$,
$\rho(\chi^{m,\epsilon})u^{m,\epsilon} \to \rho^\epsilon u^\epsilon$ *weak* in* $L^\infty(0,T;H(D))$,
$\mu(\chi^{m,\epsilon})\nabla u^{m,\epsilon} \to \mu^\epsilon\chi^\epsilon\nabla u^\epsilon$ *weakly in* $L^2(Q_T)$,
$\rho^{m,\epsilon}(u_i)_{m,\epsilon}(u_j)_{m,\epsilon} \to \rho^\epsilon u_i^\epsilon u_j^\epsilon$ *weakly in* $L^p(Q_T)$, $p > 1$.

Then passing to the limit $m \to \infty$ in (4.1) we have that $(u^\epsilon \equiv (exp(kt)u^\epsilon, \chi^\epsilon)$ satisfies

$$\int_0^T (-\rho^\epsilon u^\epsilon, \partial_t \phi) + (\mu \nabla v^\epsilon, \nabla \phi) + \tag{4.5}$$
$$(\rho^\epsilon(\tilde{u}^\epsilon + b) \cdot \nabla \phi, u^\epsilon) - (\rho^\epsilon(\tilde{u}^\epsilon + b) \cdot \nabla b, \phi) - (\rho^\epsilon v^\epsilon, \partial_t \phi))dt =$$
$$\int_0^T (f + B, \phi)dt + (\rho_0^\epsilon u_0, \phi(0)),$$
$$\partial \chi^\epsilon + \tilde{u}^\epsilon + b) \cdot \nabla \chi^\epsilon \text{ (in sense of distributions).}$$

for any $\phi \in C^1(0, T; H^1(D))$, $\nabla \cdot \phi = 0$, $\phi(T) = 0$.

To passing to the limit in (4.5) with respect to ϵ we need the convergence of $\{u^\epsilon\}$ in a suitable topology, e.g. in $L^2_{loc}(Q_T)$. For this we use the estimate

$$|\int_0^T (\partial_t(\rho^\epsilon u^\epsilon), \phi)dt| \leq c \int_0^T ||\phi||_{H^2(D)} dt$$

and obtain the convergence of the nonlinear term

$$\int_0^T (\rho^\epsilon \tilde{u}^\epsilon, u^\epsilon \cdot \nabla \phi)dt \to \int_0^T (\rho u, u \cdot \nabla \phi)dt$$

for any ϕ in $C^1((0, T); C_0^\infty(D))$. Now passing to the limit $\epsilon \to 0$ we have that (u, χ) is a weak solution of (2.4).

Theorem 1 is completely proved.

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Life span and global existence of 2-D compressible fluids

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Abstract

Consider the Euler equations of barotropic inviscid compressible fluids in the half plane. It is well known that, as the Mach number goes to zero, the compressible flows approximate the solution of the equations of motion of inviscid, incompressible fluids. In dimension 2 such limit solution is global in time. It is then natural to expect that even the compressible solution will be globally defined, for Mach numbers sufficiently small.

We decompose the solution as the sum of the irrotational part, the incompressible part and the remainder, which describes the interaction between the first two components. First we study the time behavior of the irrotational part, by means of a suitable scaling of variables. Then, we study the interaction between the two components and show the existence on any arbitrary time interval, for any Mach number sufficiently small. This yields the existence of smooth compressible flow on any arbitrary time interval. For the proofs we use a combination of energy and decay estimates.

1 Introduction

We are concerned with the existence of smooth solutions to the Euler equations for a barotropic inviscid compressible fluid in the halfplane. Let $\Omega = \mathbf{R}^2_+ = \mathbf{R}_+ \times \mathbf{R}$; we denote the boundary by $\partial\Omega = \{0\} \times \mathbf{R}$. Given $T > 0$, let us set $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$. In appropriate nondimensional form, the initial boundary value problem (ibvp) takes the form (see [5])

$$\begin{aligned}
&\partial_t \rho^\epsilon + \nabla \cdot (\rho^\epsilon v^\epsilon) = 0, && \\
&\rho^\epsilon(\partial_t v^\epsilon + (v^\epsilon \cdot \nabla) v^\epsilon) + \tfrac{1}{\epsilon^2} \nabla p^\epsilon = 0 && \text{in } Q_T, \\
&v^\epsilon \cdot \nu = 0 && \text{on } \Sigma_T, \\
&\rho^\epsilon(0,x) = \rho^\epsilon_0(x), && \\
&v^\epsilon(0,x) = v^\epsilon_0(x) && \text{in } \Omega.
\end{aligned} \tag{1.1}$$

Here the density $\rho^\epsilon = \rho^\epsilon(t,x)$, the velocity field $v^\epsilon = v^\epsilon(t,x) = (v^{\epsilon 1}, v^{\epsilon 2})$ and the pressure $p^\epsilon = p^\epsilon(t,x)$ are unknown functions of time t and space variables $x = (x_1, x_2)$. We assume that the density and the pressure are related by the equation of state $p^\epsilon = p(\rho^\epsilon)$, where $p = p(\rho)$ is a given C^5 function which satisfies $p > 0, p' > 0$ for $\rho > 0$. $\epsilon > 0$ is essentially the Mach number. We denote by ν the unit outward normal to $\partial\Omega, \partial_t = \partial/\partial t, \partial_i = \partial/\partial x_i, \nabla = (\partial_1, \partial_2), v \cdot \nabla = v^1\partial_1 + v^2\partial_2$. Solutions (ρ, v) are sought as a perturbation of the equilibrium state $(\overline{\rho}, 0)$, where $\overline{\rho} > 0$.

If $\epsilon \to 0$ and $\rho_0^\epsilon \to \overline{\rho}$, then one expects that $\rho^\epsilon \to \overline{\rho}$ and $v^\epsilon \to w$, where w is the solution to the Euler equations for incompressible flow

$$\begin{array}{ll} \nabla \cdot w = 0, & \\ \partial_t w + (w \cdot \nabla) w + \nabla \pi = 0 & \text{in } Q_T, \\ w \cdot \nu = 0 & \text{on } \Sigma_T, \\ w(0,x) = w_0(x) & \text{in } \Omega, \end{array} \tag{1.2}$$

where w_0 satisfies $\nabla \cdot w_0 = 0$ in Ω, $w_0 \cdot \nu = 0$ on $\partial\Omega$. It is well known that (1.2) may be solved globally in time, i.e. on any arbitrary time interval $[0,T]$. The aim of this paper is to solve the ibvp (1.1) on the time interval $[0,T]$, where $T > 0$ is arbitrary, for all sufficiently small ϵ and "almost" constant initial densities. The global existence of regular solutions for large data and large Mach number can't be expected in general, because the formation of singularities in finite time may occur, see [9], [6].

It is convenient to study (1.1) by making the change of variables

$$g^\epsilon = \log(\rho^\epsilon/\overline{\rho}),$$

and by introducing the function $h(s) = p'(\overline{\rho}e^s)$. Clearly, $h \in C^4(\mathbf{R}, \mathbf{R}_+)$. The system (1.1) is then equivalent to

$$\begin{array}{ll} \partial_t g^\epsilon + v^\epsilon \cdot \nabla g^\epsilon + \nabla \cdot v^\epsilon = 0, & \\ \partial_t v^\epsilon + (v^\epsilon \cdot \nabla) v^\epsilon + \frac{1}{\epsilon^2} h(g^\epsilon) \nabla g^\epsilon = 0 & \text{in } Q_T, \\ v^\epsilon \cdot \nu = 0 & \text{on } \Sigma_T, \\ g^\epsilon(0,x) = g_0^\epsilon(x), & \\ v^\epsilon(0,x) = v_0^\epsilon(x) & \text{in } \Omega, \end{array} \tag{1.3}$$

where $g_0^\epsilon(x) = \log(\rho_0^\epsilon(x)/\overline{\rho})$. Observe that $\rho_0^\epsilon \to \overline{\rho}$, $\rho^\epsilon \to \overline{\rho}$ are equivalent to $g_0^\epsilon \to 0$, $g^\epsilon \to 0$, respectively. Without loss of generality we may assume $h(0) = p'(\overline{\rho}) = 1$. It is also convenient to rescale the variables by $g(t,x) = g^\epsilon(\epsilon t, x)$, $v(t,x) = \epsilon v^\epsilon(\epsilon t, x)$ so that (1.3) becomes

$$\begin{array}{ll} \partial_t g + v \cdot \nabla g + \nabla \cdot v = 0, & \\ \partial_t v + (v \cdot \nabla) v + h(g) \nabla g = 0 & \text{in } Q_{T/\epsilon}, \\ v \cdot \nu = 0 & \text{on } \Sigma_{T/\epsilon}, \\ g(0,x) = g_0(x), & \\ v(0,x) = v_0(x) & \text{in } \Omega, \end{array} \tag{1.4}$$

where now $t \in [0, T/\epsilon]$ and $g_0 = g_0^\epsilon$, $v_0 = \epsilon v_0^\epsilon$. We will study (1.3) for small Mach number ϵ and initial density almost constant in the sense that $g_0^\epsilon = O(\epsilon)$. Accordingly, we may assume that (g_0, v_0) are $O(\epsilon)$.

We denote by $||\cdot||_m$ the norm of $H^m = H^m(\mathbf{R}^2_+)$, the Sobolev space of order $m = 1, 2, \ldots$. The norm of $L^2 = L^2(\mathbf{R}^2_+)$ is denoted by $||\cdot||$, the norm of $L^p = L^p(\mathbf{R}^2_+)$, $1 \le p \le \infty$, by $|\cdot|_p$. We will use the same symbol for spaces of vector valued functions. The norm in $H^{m-1/2}(\partial\Omega)$ is denoted by $||\cdot||_{m-1/2,\partial\Omega}$. Let X be a Banach space and $T > 0$; $C([0,T]; X)$ and $C^k([0,T]; X)$ denote respectively the spaces of continuous and k-continuously differentiable functions defined on $[0,T]$ with values in X. We define

$$X^m(T) = \bigcap_{k=0}^{m-1} C^k([0,T]; H^{m-k}), \qquad ||u||_{X^m(T)} = \sup_{[0,T]} |||u(t)|||_m$$

where

$$|||u(t)|||_m^2 = \sum_{k=0}^{m-1} ||\partial_t^k u(t)||_{m-k}^2.$$

Consider the orthogonal decomposition $L^2 = H \oplus G$ where

$$H = \{u \in L^2 | \nabla \cdot u = 0 \text{ in } \Omega, u \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad G = \{\nabla\psi | \psi \in H^1_{loc}\}.$$

Let P be the projection onto H and $Q = I - P$; it is well known that $P \in \mathcal{L}(H^m, H^m)$ for every $m \ge 0$.

We recursively define $\partial_t^k v_0$, $k \ge 1$, by formally taking $k-1$ time derivatives of (1.4), solving for $\partial_t^k v$ and evaluating it at time $t = 0$ to yield an expression in terms of the initial data (g_0, v_0). Analogously we define $\partial_t^k v_0^\epsilon$ starting from (1.3), in terms of the initial data $(g_0^\epsilon, v_0^\epsilon)$. Generic constants will be denoted by the same symbol C.

Our main result is given by the following theorem.

Theorem 1.1 *Let $T > 0$ be arbitrary. Assume $(g_0^\epsilon, v_0^\epsilon) \in H^5 \cap L^1$, and that $(g_0^\epsilon, v_0^\epsilon)$ satisfy the compatibility conditions for problem (1.3) of order 2, i.e. $\partial_t^k v_0^\epsilon \cdot \nu = 0$ on $\partial\Omega$ for $k = 0, 1, 2$; assume also that $(g_0^\epsilon, Qv_0^\epsilon)$ satisfy the compatibility conditions for problem (1.3) of order 4, i.e. $\partial_t^k Q v_0^\epsilon \cdot \nu = 0$ on $\partial\Omega$ for $k = 0, \ldots, 4$. Assume also that*

$$|g_0^\epsilon|_1 + ||g_0^\epsilon||_5 \le C_1\epsilon, \quad |v_0^\epsilon|_1 + ||v_0^\epsilon||_5 \le C_1 \qquad \forall \epsilon > 0, \tag{1.5}$$

for some constant C_1 independent of ϵ. Then there exist $\epsilon_0, C_2 > 0$ such that for all $0 < \epsilon < \epsilon_0$ the ibvp (1.3) with initial data $(g_0^\epsilon, v_0^\epsilon)$ has a unique solution $(g^\epsilon, v^\epsilon) \in X^3(T)$ and satisfies the uniform bound

$$||g^\epsilon||_{X^3(T)} \le C_2\epsilon, \quad ||v^\epsilon||_{X^3(T)} \le C_2, \qquad 0 < \epsilon < \epsilon_0. \tag{1.6}$$

The constants ϵ_0, C_2 depend on T and C_1 but are independent of ϵ.

To study the global solvability of (1.3), we decompose the solution as the sum of irrotational flow, incompressible flow and the remainder due to the nonlinear interaction. Let us consider first the irrotational component.

Given an ibvp with smooth initial data which are a small perturbation of amplitude ϵ from a constant state, the *life span* is defined as the largest time interval on which there exists a classical solution to the ibvp. From the theory of symmetric hyperbolic systems, the life span of solutions to the *Cauchy problem* is at least $O(1/\epsilon)$. Results on formation of singularities in finite time show that the life span of a classical solution is no better than $O(1/\epsilon^2)$ in 2D, [6], and $O[\exp(1/\epsilon^2)]$ in 3D, [9]. Under suitable assumptions for the initial data, the length of the life span can be extended. Alinhac [1], [2] and Sideris [10] show that the life span of 2D rotationally symmetric compressible flow is $O(1/\epsilon^2)$. The chief techniques employed in [2], [10] are energy and decay estimates using vector fields associated to the natural invariance of the equations.

We extend the results of Alinhac and Sideris to the case of the halfplane. We consider smooth irrotational solutions to (1.4) and show that their life span is at least $O(1/\epsilon^\alpha)$, for a suitable α, $1 < \alpha < 2$, see Theorem 3.1. In this case the approach of [1], [2] and [10] doesn't seem to work because of the boundary. However, the result is again obtained by a combination of energy and decay estimates. The main tool for the estimation of the solution (g, v) is a new energy estimate for the solution of the wave-type equation (2.2). Moreover, we apply the decay estimate of [4] for the solution to the equations of linearized acoustics.

The estimate of the life span yields the existence of the irrotational component in the arbitrary interval $[0, T]$, for all sufficiently small ϵ. By well known results, the incompressible component is also globally defined on $[0, T]$. The global solvability of the whole solution follows by showing the existence of the non linear interaction up to time T. For, we use in particular the decay in time of the irrotational component and the scaling properties of the incompressible one.

The complete proofs of Theorems 1.1 and 3.1 are given in [8]; a similar estimate of the life span of irrotational solutions was previously shown in [7].

2 A second order hyperbolic equation

We assume that the solution of (1.4) satisfies

$$(g,\, v) \in X^5(T), \quad 0 < m^{-1} \le h \le M \quad \text{on } Q_T. \tag{2.1}$$

The assumption $h(0) = 1$ yields $m, M \ge 1$. We apply the operator $\partial_t + v \cdot \nabla$ to the first equation of (1.4), take the divergence of the second equation, then the difference and obtain the wave-type equation

$$(\partial_t + v \cdot \nabla)^2 g - \nabla \cdot (h(g)\nabla g) = \sum_{i,j=1,2} \partial_i v^j \partial_j v^i. \tag{2.2}$$

The velocity equation and the boundary condition on v yield the homogeneous Neumann condition

$$\partial_1 g = 0 \qquad \text{on } \Sigma_T. \tag{2.3}$$

Finally we add the initial conditions

$$g(0,x) = g_0(x), \qquad \partial_t g(0,x) = g_1(x) \qquad \text{in } \Omega, \tag{2.4}$$

where $g_1 = -(v_0 \cdot \nabla g_0 + \nabla \cdot v_0)$. The problem (2.2), (2.3), (2.4) is a well-known tool in the study of the compressible Euler equations in domains with boundary, see [3] and references thereinto. Our aim is to establish the new estimates (2.10) and (2.12) which allow to take account of the pointwise decay of the solution.

We start with the study of the linear hyperbolic mixed problem

$$\begin{array}{lll} (\partial_t + v\cdot\nabla)^2 G - \nabla\cdot(h(g)\nabla G) = F & & \text{in } Q_T, \\ \partial_1 G = 0 & & \text{on } \Sigma_T, \\ G(0,x) = G_0(x), & \partial_t G(0,x) = G_1(x) & \text{in } \Omega, \end{array} \tag{2.5}$$

for some given F, G_0, G_1. Let us set $B = \partial_t + v\cdot\nabla, \int = \int_\Omega dx$,

$$E(G)(t) = (\int |BG(t)|^2 + \int h|\nabla G|^2)^{1/2}.$$

We first prove an apriori estimate of $E(G)$ for solutions of (2.5) and solutions of the Cauchy-Dirichlet problem

$$\begin{array}{lll} (\partial_t + v\cdot\nabla)^2 G - \nabla\cdot(h(g)\nabla G) = F & & \text{in } Q_T, \\ G = 0 & & \text{on } \Sigma_T, \\ G(0,x) = G_0(x), & \partial_t G(0,x) = G_1(x) & \text{in } \Omega. \end{array} \tag{2.6}$$

Since on $\partial\Omega$ either $G = 0$, which yields $BG = 0$, or $\partial_1 G = 0$, we can easily show

Lemma 2.1 *Let the assumption (2.1) hold and let G be a sufficiently smooth solution of either (2.5) or (2.6). Then the estimate*

$$\frac{d}{dt}E(G) \le C\Phi E(G) + ||F||$$

holds in $[0,T]$, *where* $\Phi = |\nabla v|_\infty + |\partial_t(mh)|_\infty + |\nabla\cdot(mhv)|_\infty$.

Let us introduce the notation $g_t = \partial_t g, F_t = \partial_t F$ and so on, where now $F = \sum \partial_i v^j \partial_j v^i$. By time differentiation of (2.2) we have

$$\begin{array}{ll} B^2 g_t - \nabla\cdot(h\nabla g_t) = F_{(t)} & \text{in } Q_T, \\ \partial_1 g_t = 0 & \text{on } \Sigma_T, \end{array} \tag{2.7}$$

where

$$F_{(t)} = F_t - B(v_t\cdot\nabla g) - v_t\cdot\nabla Bg + \nabla\cdot(h_t\nabla g).$$

Let us denote by g_x either $\partial_1 g$ or $\partial_2 g$ and analogously v_x and so on. By differentiation of (2.2) we have

$$B^2 g_x - \nabla\cdot(h\nabla g_x) = F_{(x)} \qquad \text{in } Q_T, \tag{2.8}$$

where

$$F_{(x)} = F_x - B(v_x\cdot\nabla g) - v_x\cdot\nabla Bg + \nabla\cdot(h_x\nabla g).$$

If $g_x = \partial_1 g$, we add to (2.8) the Dirichlet condition $g_x = 0$ on Σ_T. If $g_x = \partial_2 g$, we add the Neumann condition $\partial_1 g_x = 0$ on Σ_T, obtained by differentiation of (2.3) along

the boundary. We proceed similarly for the derivatives g_{tt}, g_{tx}, g_{ttt}, g_{ttx}, g_{tttt}, g_{tttx}. They satisfy an equation of the previous form with right hand side respectively denoted by $F_{(tt)}$, $F_{(tx)}, \ldots, F_{(tttt)}$, $F_{(tttx)}$. Since each one of the above derivatives solves a problem of the form (2.5) or (2.6) with either Dirichlet or Neumann boundary conditions, we can apply Lemma 2.1 to each problem and, combining with the estimate for g itself, obtain the following result. Let us define

$$Z = E(g) + E(g_t) + E(g_x) + \ldots + E(g_{tttt}) + E(g_{tttx}),$$

$$\Lambda = ||F|| + ||F_{(t)}|| + ||F_{(x)}|| + \ldots + ||F_{(tttt)}|| + ||F_{(tttx)}||.$$

Lemma 2.2 *Let g be a sufficiently smooth solution of (2.2) - (2.4) satisfying (2.1). Then the estimate*

$$\frac{d}{dt}Z(t) \leq C\Phi Z(t) + \Lambda(t) \tag{2.9}$$

holds in $[0, T]$.

The next step is the estimate of $\Lambda(t)$. Let us set for convenience

$$\chi \equiv |u|_\infty^{3/4}||\nabla u||_4^{1/4}.$$

Lemma 2.3 *Let g be a sufficiently smooth solution of (2.2) - (2.4) satisfying (2.1). Then the estimate*

$$\tfrac{d}{dt}Z \leq K_1(1 + |u|_\infty)\chi Z + K_1(1 + |u|_\infty)^9\,(|u|_\infty + \chi)||\nabla u||_4 \tag{2.10}$$

holds in $[0, T]$, where K_1 denotes a positive constant which may depend boundedly on $h(g)$ and its derivatives up to order 4.

Proof. We write explicitly all the right hand sides $F, \ldots, F_{(tttx)}$ and estimate them by the Hölder inequality and suitable interpolation inequalities. We obtain

$$\Lambda \leq K(1 + |u|_\infty)^9\,(|u|_\infty + \chi)||\nabla u||_4 \tag{2.11}$$

in $[0, T]$, where K denotes from now on a positive constant which may depend boundedly on $h(g)$ and its derivatives up to order 4. Moreover we have $\Phi \leq K(1 + |u|_\infty)\chi$. Thus from (2.9) and (2.11) we have the thesis. ∎

Finally, we estimate $||\nabla u||_4$ in terms of $|u|_\infty$ and Z. Let us set $\operatorname{rot} v = \partial_1 v^2 - \partial_2 v^1$.

Lemma 2.4 *Let $u = (g, v)$ be a sufficiently smooth solution of (1.4) satisfying (2.1). Then the estimate*

$$||\nabla u||_4 \leq K_2\{Z + ||g|| + ||rot v||_4 + (1 + |u|_\infty)^6|u|_\infty||\nabla u||_4\} \tag{2.12}$$

holds in $[0, T]$, where K_2 denotes a positive constant which may depend boundedly on $h(g)$ and its derivatives up to order 4.

Proof. The estimate of v follows from the well known elliptic estimate

$$||\nabla v||_m \leq C(||\mathrm{rot}\, v||_m + ||\nabla \cdot v||_m + ||v \cdot \nu||_{m+1/2,\partial\Omega}), \tag{2.13}$$

and from $\nabla \cdot v = -Bg$ and interpolation inequalities. To estimate g, we consider (2.2), (2.3) written in the form of the elliptic problem

$$-\Delta g = \frac{1}{h}(\nabla h \cdot \nabla g - B^2 g + F) \quad \text{in } \Omega, \qquad \partial_1 g = 0 \quad \text{on } \partial\Omega. \quad \blacksquare$$

3 Life span of irrotational solutions

The aim of this section is the proof of the following theorem. For more details see [8]; a similar result was previously shown in [7].

Theorem 3.1 *Assume $(g_0, v_0) \in H^5 \cap L^1$, where $Pv_0 = 0$, and that the data satisfy the (necessary) compatibility conditions of order 4 for problem (1.4), i.e. $\partial_t^k v_0 \cdot \nu = 0$ on $\partial\Omega$ for $k = 0, \ldots, 4$. Assume also that*

$$|(g_0, v_0)|_1 + ||(g_0, v_0)||_5 \leq C_3\epsilon \qquad \forall \epsilon > 0, \tag{3.1}$$

for some constant C_3 independent of ϵ. Then there exist constants $\delta, \epsilon_0, A, C_4 > 0$, where $1/4 \leq \delta < 1/2$, such that for all $0 < \epsilon < \epsilon_0$ the ibvp (1.4) has a unique solution $(g, v) \in X^5(A/\epsilon^\alpha)$, where $\alpha = \frac{3+8\delta}{3+5\delta}$, which satisfies

$$||(g, v)||_{X^5(A/\epsilon^\alpha)} \leq C_4\epsilon, \qquad 0 < \epsilon < \epsilon_0, \tag{3.2}$$

and

$$Pv(t) = 0, \qquad 0 < t < A/\epsilon^\alpha.$$

The constants ϵ_0, A depend decreasingly on C_3; C_4 depends increasingly.

Proof. We begin by observing that irrotational initial velocities are propagated, i.e. if $Pv_0 = 0$, then $Pv(t) = 0$ for $t > 0$, as long a smooth solution exists.

For the reader's convenience, we recall the decay estimate of Iguchi [4]. Let us introduce the linearized operator of acoustics

$$L = -i \begin{pmatrix} 0 & \nabla \cdot \\ \nabla & 0 \end{pmatrix},$$

$$D(L) = \{u = (g, v) \in L^2 \,|\nabla g \in L^2, \nabla \cdot v \in L^2, v \cdot \nu = 0 \text{ on } \partial\Omega\}.$$

L is self-adjoint. Let us denote by $N(L)$ the null space of L in L^2, and by Γ_0 and Γ the orthogonal projections in L^2 onto $N(L)$ and $N(L)^\perp$, respectively. The Helmholtz decomposition gives $N(L) = \{0\} \times H, N(L)^\perp = L^2 \times G$; moreover $\Gamma_0(u) = (0, Pv)$ and $\Gamma(u) = (g, Qv)$. Iguchi [4] shows the following result.

Lemma 3.1 *Let $n \geq 2$ and $l > n/2$ not necessarily integer. Then, there exist constants $C > 0$ and $0 < \delta < \frac{1}{2}$ such that*

$$|\Gamma e^{-iLt}u_0|_\infty \leq C|t|^{-\delta}|u_0|_1^{2\delta}||u_0||_l^{1-2\delta} \tag{3.3}$$

holds for any $t \in \mathbf{R}, t \neq 0, u_0 = (g_0, v_0) \in H^l(\mathbf{R}^n_+) \cap L^1(\mathbf{R}^n_+)$ with $v_0 \cdot \nu \in H^1_0(\mathbf{R}^n_+)$. Moreover,

$$|\Gamma e^{-iLt}u_0|_\infty \leq C||u_0||_l \tag{3.4}$$

holds for any $t \in \mathbf{R}, u_0 \in H^l(\mathbf{R}^n_+)$ with $v_0 \cdot \nu \in H^1_0(\mathbf{R}^n_+)$.

Let us write (1.4) as $u_t + iLu = \mathcal{F}, u(0) = u_0$, where $\mathcal{F} = (-v \cdot \nabla g, -(v \cdot \nabla)v + (1-h)\nabla g)$, $u_0 = (g_0, v_0)$. We transform it into the integral equation

$$u(t) = e^{-iLt}u_0 + \int_0^t e^{-iL(t-\tau)}\mathcal{F}(\tau)d\tau. \tag{3.5}$$

The evolution operator e^{-iLt} is decomposed as $e^{-iLt} = \Gamma_0 e^{-iLt} + \Gamma e^{-iLt} = \Gamma_0 + \Gamma e^{-iLt}$. Since $Pv_0 = 0$ and $Pv = 0$, we have $\Gamma_0 u_0 = 0$, $\Gamma_0\mathcal{F} = 0$. Let us choose $l = 4$; (3.3) holds with $\delta = 1/4$, see [4]. Applying (3.3), (3.4) to the solution $u(t)$ of (3.5) and estimating $\mathcal{F}$ we obtain

$$\begin{aligned} &|u(t)|_\infty \leq K_3(1+t)^{-\delta}(||u_0||_4^2 + |u_0|_1^2)^\delta ||u_0||_4^{1-2\delta} \\ &+K_3 \textstyle\int_0^t (t-\tau)^{-\delta}(1+||u(\tau)||_2)^{3(1-2\delta)}|u(\tau)|_\infty^{1-2\delta}||u(\tau)||^{2\delta}||\nabla u(\tau)||_4 d\tau \end{aligned} \tag{3.6}$$

for a suitable K_3. From now on we assume $0 < \epsilon < 1$. Let us define $U(t)$ by $U^2(t) = \int(g^2 + \frac{1}{h}|v|^2)(t)$. From (3.1) we can find constants $k_0, K_0 > 0$ such that

$$K_3(||u_0||_4^2 + |u_0|_1^2)^\delta ||u_0||_4^{1-2\delta} \leq k_0\epsilon, \qquad U(0) + Z(0) \leq K_0\epsilon.$$

Let us set $K_4 = 2K_0(M + 2K_2)$. If u would satisfy $||u(t)||_5 \leq K_4\epsilon$ in $[0,T]$, from (3.6) we would obtain

$$\begin{aligned} &|u(t)|_\infty \leq k_0\epsilon(1+t)^{-\delta} \\ &+K_3K_4^{1+2\delta}(1+K_4)^{3(1-2\delta)}\epsilon^{1+2\delta}\textstyle\int_0^t (t-\tau)^{-\delta}|u(\tau)|_\infty^{1-2\delta}d\tau. \end{aligned}$$

Then Lemma 5.1 would imply

$$|u(t)|_\infty \leq 2k_0\epsilon(1+t)^{-\delta} + C_0\epsilon^{1+\frac{1}{2\delta}}t^{\frac{1-\delta}{2\delta}}, \qquad t \geq 0, \tag{3.7}$$

for a suitable constant $C_0 = C_0(K_3, K_4)$. Let us define

$$\begin{aligned} T_1 = \sup\{T > 0 : &\ U(t) + Z(t) \leq 2K_0\epsilon, \\ &|u(t)|_\infty \leq 2k_0\epsilon(1+t)^{-\delta} + C_0\epsilon^{1+\frac{1}{2\delta}}t^{\frac{1-\delta}{2\delta}} \text{ if } 0 < t < T\}, \end{aligned} \tag{3.8}$$

$$T_2 = \min\{T_1, \tfrac{A}{\epsilon^\alpha}\},$$

where $A \le 1$ will be fixed later and $\alpha = \frac{3+8\delta}{3+5\delta}$. Given any T, $0 < T \le T_2$, we have $\epsilon^{1+\frac{1}{2\delta}}t^{\frac{1-\delta}{2\delta}} \le A^{\frac{1-\delta}{2\delta}}\epsilon^{\frac{3+9\delta}{3+5\delta}} \le \epsilon$, for any $t \in [0,T]$. It follows from the estimate in (3.8) that $|u(t)|_\infty$ is uniformly bounded in $[0,T]$ by $(2k_0 + C_0)\epsilon$; this implies the boundedness of m, M and all constants of K-type. If $0 < \epsilon < \epsilon_0$, where ϵ_0 is small enough, we show from (2.12) that u satisfies

$$||\nabla u||_4 \le 4K_0K_2\epsilon. \tag{3.9}$$

We multiply $(1.4)_1$ by g, $(1.4)_2$ by v/h and integrate over Ω, then we apply the Gronwall lemma and estimate the right hand side. A small enough yields

$$||g||^2 + ||v||^2 \le 2MK_0^2\epsilon^2.$$

From this estimate plus (3.9) we have $||u||_5 \le K_4\epsilon$, and consequently the estimate concerning $|u(t)|_\infty$ in (3.8), provided $0 \le t \le T \le T_2$ and $0 < \epsilon < \epsilon_0$, as already said. We apply the Gronwall lemma to (2.10), estimate the right hand side and restrict A further, if necessary, in such a way that $U(t) + Z(t) < 2K_0\epsilon$. This shows $A/\epsilon^\alpha \le T_1$, i.e. the life span is at least A/ϵ^α.

Corollary 3.1 *Let $u = (g,v)$ be the solution constructed in Theorem 3.1. Then*

$$|u(t)|_\infty \le C\,\epsilon(1+t)^{-4\delta/(3+8\delta)},$$

for $0 < t \le A/\epsilon^\alpha$ and for a suitable constant C.

4 Proof of Theorem 1.1

Take an arbitrary $T > 0$ and initial data $(g_0^\epsilon, v_0^\epsilon)$ that satisfy the assumptions of Theorem 1.1. Let us consider $g_0 = g_0^\epsilon, v_0 = \epsilon v_0^\epsilon$. We readily show that (g_0, v_0) satisfy the compatibility conditions of order 2 and (g_0, Qv_0) the compatibility conditions of order 4, both for problem (1.4). Moreover (g_0, v_0) satisfy (3.1). Let $u_1 = (g_1, v_1)$ be the solution of (1.4) with irrotational initial data (g_0, Qv_0). By the results of Theorem 3.1 and its corollary there are $0 < \epsilon_0 < 1, A > 0, 1/4 \le \delta < 1/2$, such that

$$|||u_1(t)|||_5 \le C\epsilon, \qquad |u_1(t)|_\infty \le C\epsilon(1+t)^{-4\delta/(3+8\delta)}, \tag{4.1}$$

for all $0 \le t \le A/\epsilon^\alpha$, $0 < \epsilon < \epsilon_0$, where $\alpha = \frac{3+8\delta}{3+5\delta} > 1$. By some interpolation inequalities, it follows from (4.1) that there exist $C > 0, 0 < \delta' < 1/2$ such that

$$\begin{aligned} &\textstyle\sum_{k=0}^1(|D^{1-k}\partial_t^k u_1|_4 + |D^{1-k}\partial_t^k u_1|_8) + \sum_{k=0}^2 |D^{2-k}\partial_t^k u_1|_4 \\ &+ \textstyle\sum_{k=0}^3 |D^{3-k}\partial_t^k u_1|_{8/3} + \sum_{k=0}^4 |D^{4-k}\partial_t^k u_1|_{5/2} \le C\epsilon(1+t)^{-\delta'}, \end{aligned} \tag{4.2}$$

for all $0 \le t \le A/\epsilon^\alpha$, $0 < \epsilon < \epsilon_0$. We restrict further ϵ_0, if necessary, in such a way that $A/\epsilon_0^{\alpha-1} > T$. It follows that u_1 is defined up to time T/ϵ.

Since the initial data are uniformly bounded in $L^1 \cap H^5$, for any ϵ there exists a globally defined solution of (1.2) $(\pi^\epsilon, w^\epsilon) \in X^5(T)$ with initial velocity Pv_0^ϵ. Moreover there is a constant $\hat{C}$ depending on T such that $||(\pi^\epsilon, w^\epsilon)||_{X^5(T)} \le \hat{C}$. Define

$\pi(t,x) = \epsilon^2\pi^\epsilon(\epsilon t,x), w(t,x) = \epsilon w^\epsilon(\epsilon t,x)$. Then $(\pi,w) \in X^5(T/\epsilon)$ solves (1.2) with initial data $Pv_0 = \epsilon P v_0^\epsilon$ and

$$||\partial_t^k \pi(t)||_{5-k} \leq \hat{C}\epsilon^{k+2}, \qquad ||\partial_t^k w(t)||_{5-k} \leq \hat{C}\epsilon^{k+1}, \tag{4.3}$$

for $0 \leq t \leq T/\epsilon$ and $k = 0,\dots,4$. As in [10] we seek a solution of (1.4) in the form

$$g = g_1 + \pi + g_2, \qquad v = v_1 + w + v_2.$$

In order that (g,v) solve (1.4), the corrections (g_2, v_2) should satisfy

$$\begin{array}{ll} \partial_t g_2 + v\cdot\nabla g_2 + \nabla\cdot v_2 = F_1, & \\ \partial_t v_2 + (v\cdot\nabla)v_2 + h(g)\nabla g_2 = F_2 & \text{in } Q_{T/\epsilon}, \\ v_2\cdot\nu = 0 & \text{on } \Sigma_{T/\epsilon}, \\ g_2(0,x) = -\pi(0,x), & \\ v_2(0,x) = 0 & \text{in } \Omega, \end{array} \tag{4.4}$$

where

$$F_1 = v_1\cdot\nabla g_1 - v\cdot\nabla(g_1+\pi) - \partial_t\pi,$$

$$F_2 = (v_1\cdot\nabla)v_1 + (w\cdot\nabla)w - (v\cdot\nabla)(v_1+w)$$
$$+[h(g_1) - h(g)]\nabla g_1 + [1-h(g)]\nabla\pi.$$

Note that $||g_2(0)||_3 = ||\pi(0)||_3 \leq \hat{C}\epsilon^2$. We are going to construct a solution of (4.4) in the space $X^3(T/\epsilon)$; it will be $O(\epsilon^\mu)$ with a suitable $\mu > 1$.

Let $u_2 = (g_2, v_2)$ be a sufficiently smooth solution of (4.4) in $[0,T']$, where $T' \leq T/\epsilon$. We apply $B = \partial_t + v\cdot\nabla$ to the first equation of (4.4), $\nabla\cdot$ to the second one and take account of the boundary conditions. We obtain the mixed problem

$$\begin{array}{ll} B^2 g_2 - \nabla\cdot(h(g)\nabla g_2) = H & \text{in } Q_{T/\epsilon}, \\ \partial_1 g_2 = 0 & \text{on } \Sigma_{T/\epsilon}, \\ g_2(0,x) = -\pi(0,x) & \text{in } \Omega, \\ \partial_t g_2(0,x) = v_0\cdot\nabla\pi(0,x) + F_1(0,x) & \text{in } \Omega, \end{array} \tag{4.5}$$

where

$$H = BF_1 - \nabla\cdot F_2 + \sum_{i,j}\partial_i v^j \partial_j v_2^i.$$

We apply to (4.5) the same arguments leading to Lemma 2.2. The derivatives $g_{2t} = \partial_t g_2, g_{2x} = \partial_x g_2, g_{2tt} = \partial_t^2 g_2, g_{2tx} = \partial_x\partial_t g_2$ satisfy the wave-type equation in (4.5) with right-hand side respectively denoted by $H_{(t)}, H_{(x)}, H_{(tt)}, H_{(tx)}$. Let us define

$$Z' = E(g_2) + E(g_{2t}) + E(g_{2x}) + E(g_{2tt}) + E(g_{2tx}),$$

$$\Lambda' = ||H|| + ||H_{(t)}|| + ||H_{(x)}|| + ||H_{(tt)}|| + ||H_{(tx)}||.$$

Lemma 4.1 *Let g_2 be a sufficiently smooth solution of (4.5) in the time interval $[0,T']$, where $T' \le T/\epsilon$. Then the estimate*

$$\frac{d}{dt}Z'(t) \le C\Phi Z'(t) + \Lambda'(t)$$

holds in $[0,T']$, where $\Phi = |\nabla v|_\infty + |\partial_t(mh(g))|_\infty + |\nabla\cdot(mh(g)v)|_\infty$.

From $||g_2(0)||_3 = ||\pi(0)||_3 \le \hat{C}\epsilon^2, v_2(0) = 0$, and (4.4), there exists a constant $K_5 > 0$ such that $|||u_2(0)|||_3 \le K_5\epsilon^2$. Choose $1 < \mu < 1+\delta'$ and define

$$T_3 = \sup\{T' > 0 : |||u_2(t)|||_3 \le 2K_5\epsilon^\mu \quad \text{if } 0 < t < T'\},$$

$$T_4 = \min\{T_3, \frac{T}{\epsilon}\}, \qquad \zeta = \operatorname{rot} v_2.$$

Lemma 4.2 *Let $u_2 = (g_2, v_2)$ be a sufficiently smooth solution of (4.4) in the time interval $[0,T']$, where $0 < T' \le T_4$. Then there exist $\epsilon_0, K > 0$ such that*

$$||\nabla u_2||_2 + ||\partial_t u_2||_2 + ||\partial_t^2 u_2||_1 \le K(||\zeta||_2 + Z' + ||g_2|| + \epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu}) \quad (4.6)$$

holds in $[0,T']$, for all $0 < \epsilon < \epsilon_0$.

Proof. The estimates of ∇v_2, $\partial_t v_2$ and $\partial_t^2 v_2$ follow from (2.13) and (4.2) - (4.4). Moreover, g_2 is estimated by considering (4.5) written as the elliptic problem

$$-\Delta g_2 = \frac{1}{h}(\nabla h\cdot\nabla g_2 - B^2 g_2 + H) \quad \text{in } \Omega, \qquad \partial_1 g_2 = 0 \quad \text{on } \partial\Omega. \qquad \blacksquare$$

Lemma 4.3 *Let u_2 be a sufficiently smooth solution of (4.4) in the time interval $[0,T']$, where $0 < T' \le T_4$. Then there exist $\epsilon_0, K > 0$ such that*

$$\frac{d}{dt}Z' \le K\{(\Phi+\epsilon)Z' + \epsilon||g_2|| + \epsilon||\zeta||_2 + \epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu}\}. \quad (4.7)$$

holds in $[0,T']$, for all $0 < \epsilon < \epsilon_0$.

Proof. It follows from Lemmata 4.1, 4.2 and the estimate

$$\begin{aligned} ||H|| + ||H_{(t)}|| + ||H_{(x)}|| + ||H_{(tt)}|| + ||H_{(tx)}|| &\le K\{\epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu} \\ &+\epsilon||\nabla v_2||_1 + \epsilon||\nabla\partial_t v_2||_1 + \epsilon||\nabla\partial_t^2 v_2|| + \epsilon||\nabla g_{2t}|| + \epsilon||\nabla g_{2tt}||\}. \end{aligned} \quad (4.8) \qquad \blacksquare$$

Lemma 4.4 *Let u_2 be a sufficiently smooth solution of (4.4) in the time interval $[0,T']$, where $0 < T' \le T_4$. Then there exist $\epsilon_0, K > 0$ such that*

$$\tfrac{d}{dt}||\zeta||_2 \le K\{\epsilon||\zeta||_2 + \epsilon Z' + \epsilon||g_2|| + \epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu}\}. \quad (4.9)$$

holds in $[0,T']$, for all $0 < \epsilon < \epsilon_0$.

Proof. Apply the operator *rot* to the velocity equation in (4.4). By standard devices it follows that

$$\frac{d}{dt}||\zeta||_2 \leq K\{\epsilon||\nabla v_2||_2 + \epsilon||\nabla g_2||_2 + \epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu}\}.$$

Finally, take account of (4.6). ∎

Lemma 4.5 *Let u_2 be a sufficiently smooth solution of (4.4) in the time interval $[0,T']$, where $0 < T' \leq T_4$. Then there exist $\epsilon_0, K > 0$ such that*

$$\frac{d}{dt}\Big(\int g_2^2 + \frac{1}{h}|v_2|^2\Big)^{1/2} \leq K\{\epsilon\Big(\int g_2^2 + \frac{1}{h}|v_2|^2\Big)^{1/2} + \epsilon^2(1+t)^{-\delta'} + \epsilon^3\}. \tag{4.10}$$

holds in $[0,T']$, for all $0 < \epsilon < \epsilon_0$.

Proof. It follows by multiplying $(4.4)_1$ by g_2, $(4.4)_2$ by v_2/h and integration on Ω. ∎

We add (4.7), (4.9) and (4.10). Since we can show that $\Phi \leq K\epsilon$, we obtain

$$\begin{aligned}&\tfrac{d}{dt}\{Z' + ||\zeta||_2 + (\textstyle\int g_2^2 + \frac{1}{h}|v_2|^2)^{1/2}\}\\ &\leq K_7\{\epsilon||\zeta||_2 + \epsilon Z' + \epsilon(\textstyle\int g_2^2 + \frac{1}{h}|v_2|^2)^{1/2} + \epsilon^2(1+t)^{-\delta'} + \epsilon^{2\mu}\},\end{aligned} \tag{4.11}$$

for a suitable constant $K_7 > 0$. There exists $K_8 > 0$ such that

$$Z'(0) + ||\zeta(0)||_2 + \Big(\int g_2^2(0) + \frac{1}{h}|v_2(0)|^2\Big)^{1/2} \leq K_8\epsilon^2; \tag{4.12}$$

moreover, from (4.6) there exists $K_9 > 0$ such that

$$|||u_2(t)|||_3 \leq K_9\{Z'(t) + ||\zeta(t)||_2 + \Big(\int g_2^2(t) + \frac{1}{h}|v_2(t)|^2\Big)^{1/2} + \epsilon^2\} \tag{4.13}$$

for each $t \in [0,T']$. We apply the Gronwall lemma to (4.11), substitute (4.12), (4.13) and restrict ϵ_0 further, if necessary, in such a way that $|||u_2(t)|||_3 < 2K_5\epsilon^\mu$. This shows that $T/\epsilon \leq T_3$, i.e. (g_2, v_2) exist up to time T/ϵ. Consequently, also $g = g_1 + \pi + g_2, v = v_1 + w + v_2$ exist up to time T/ϵ. Finally, we define rescaled variables by

$$g^\epsilon(t,x) = g(\frac{t}{\epsilon}, x), \qquad v^\epsilon(t,x) = \frac{1}{\epsilon}v(\frac{t}{\epsilon}, x).$$

Then (g^ϵ, v^ϵ) is defined up to time T, lies in the space $X^3(T)$ and is the solution of (1.3). From (4.1), (4.3) and $|||u_2(t)|||_3 \leq 2K_5\epsilon^\mu$ one obtains (1.6).

5 Appendix

The next lemma gives the result needed for (3.7); for the proof see [7].

Lemma 5.1 *Let δ be a constant such that $0 < \delta < 1/2$. Let $\varphi : \mathbf{R}_+ \mapsto \mathbf{R}_+$ be a continuous positive function satisfying the inequality*

$$\varphi(t) \leq k_0\epsilon(1+t)^{-\delta} + C\epsilon^{1+2\delta} \int_0^t (t-\tau)^{-\delta} \varphi(\tau)^{1-2\delta} d\tau$$

for $t \geq 0$, $\epsilon > 0$ and some constants k_0, C independent of ϵ. Then

$$\varphi(t) \leq 2k_0\epsilon(1+t)^{-\delta} + C_0\epsilon^{1+\frac{1}{2\delta}} t^{\frac{1-\delta}{2\delta}}, \qquad t \geq 0,$$

for a suitable constant C_0 which depends on k_0 and C.

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On the theory of nonstationary hydrodynamic potentials

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1 Introduction

In the present paper we consider initial-boundary value problem for the Stokes equations

$$\vec{v}_t - \Delta\vec{v} + \nabla p = 0, \qquad \nabla \cdot \vec{v} = 0, \qquad x \in \Omega,\ t \in (0,T),$$
$$\vec{v}|_{t=0} = \vec{v}_0(x), \qquad \vec{v}|_S = \vec{a}(x',t) \tag{1.1}$$

in a bounded convex domain $\Omega \in R^n$, $n \geq 2$, with a smooth boundary S on the basis of the theory of nonstationary hydrodynamic potentials constructed by J.Leray [13] in the case of two and by K.K.Golovkin [2,5] in the case of three spatial variables. A central role in the theory is played by a special fundamental solution of the Stokes system which permits to obtain an explicit solution of the problem (1.1) with $\vec{v}_0(x) = 0$ and with $\Omega = R^n_+ = \{x_n > 0\}$. For $n = 2$, it was found by C.W.Oseen [14] in the form of an analytic function of $z = x_1 + ix_2$ defined in the half-plane $x_2 > 0$ and used by J.Leray as a kernel of his potentials. In the three-dimensional case such a solution was constructed in [2,15].

With this approach, the problem (1.1) with zero initial data reduces to a system of integral equations of the Volterra-Fredholm type which is solvable under different hypotheses concerning $\vec{a}(x,t)$. In particular, K.K.Golovkin has considered the case of continuous $\vec{a}(x,t)$ satisfying the Hölder condition with respect to the spatial variables. His estimates enabled him to analyze the problem (1.1) with $\vec{a} = 0$, $\vec{v}_0 \in L_p(\Omega)$. Moreover, he has studied a nonlinear problem with $\vec{v}_0 \in L_p(\Omega)$, $p > 3$.

We restrict ourselves to the case $\vec{a}(x,t) \cdot \vec{n}(x) = 0$ where $\vec{n}$ is a unit interior normal to S. Our main result is as follows.

Theorem 1.1. *Assume that $S \in C^{2+\alpha}$, $\alpha \in (0,1)$. For arbitrary $\vec{a}(x,t)$ and $\vec{v}_0(x)$ which are continuous and satisfy the compatibility conditions*

$$\vec{a}(x,0) = \vec{v}_0(x)|_S, \qquad \nabla \cdot \vec{v}_0(x) = 0,$$

and the condition $\vec{a}(x,t) \cdot \vec{n}(x) = 0$, the problem (1.1) has a solution with $\vec{v}(x,t)$ continuous in $Q_T = \Omega \times (0,T)$ and satisfying the inequality

$$\sup_{x\in\Omega} \sup_{t<T} |\vec{v}(x,t)| \le c(T)\Big(\sup_{x\in S} \sup_{t<T} |\vec{a}(x,t)| + \sup_{x\in\Omega} |\vec{v}_0(x)|\Big). \tag{1.2}$$

By another method the estimate (1.2) in the case $\vec{a} = 0$ was obtained in the paper [16]. Unfortunately, the detailed proof of this result was not published. Our arguments here are considerably simpler.

In Section 1 we consider the problem (1.1) with zero initial data by means of hydrodynamic potentials assuming only that $S \in C^{1+\gamma}$, $\gamma \in (0,1)$. Here we follow the arguments of J.Leray and K.K.Golovkin especially closely. In Section 2 we take account of non-zero initial state and complete the proof of Theorem 1.1. We use the condition $S \in C^{2+\alpha}$ but it is clear that this requirement can be weakened.

K.K.Golovkin (1936-1969) has started working on the theory of hydrodynamic potentials when he was a student of the Leningrad University. The paper [1] where he completed J.Leray's results was his master thesis. In addition to many results in the theory of partial differential equations (see for instance [1,2,5,8,10]), he is known due to his important contribution to the theory of functions, namely, his theory of a large class of so called fractional spaces [4,6,7,9]. He gave an elegant method of estimation of convolution integrals in these spaces [3]. In this year he would be 65, and the present paper is devoted to his memory.

2 The second fundamental solution and the problem with zero initial data

The solution of the initial-boundary value problem in the half-space $R^n_+ = \{x_n > 0\}$

$$\vec{v}_t - \Delta\vec{v} + \nabla p = 0, \qquad \nabla \cdot \vec{v} = 0, \qquad x \in R^n_+,\ t > 0,$$

$$\vec{v}|_{t=0} = 0, \qquad \vec{v}|_{x_n=0} = \vec{a}(x',t) \tag{2.1}$$

where $x' = (x_1, ..., x_{n-1})$ and $a_n(x',t) = 0$ is expressed as the sum of potentials

$$v_i(x,t) = \sum_{j=1}^{n-1} \int_0^t \int_{R^{n-1}} G^{(0)}_{ij}(x'-y', x_n, t-\tau) a_j(y',\tau)\,dy'd\tau, \tag{2.2}$$

$$p(x,t) = -4\Big(\frac{\partial}{\partial t} - \Delta\Big) \sum_{j=1}^{n-1} \int_0^t \int_{R^{n-1}} Q^{(0)}_j(x'-y', x_n, t-\tau) a_j(y',\tau)\,dy'd\tau. \tag{2.3}$$

Here

$$G_{ij}^{(0)}(x,t) = -2\frac{\partial\Gamma(x,t)}{\partial x_n}\delta_{ij} + 4\frac{\partial}{\partial x_j}Q_i^{(0)}(x,t), \qquad i = 1,...,n,\ j = 1,...,n-1,$$

$$Q_i^{(0)}(x,t) = \int_0^{x_n} dy_n \int_{R^{n-1}} \frac{\partial\Gamma(y,t)}{\partial y_n}\frac{\partial E(x-y)}{\partial x_i}dy',$$

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}} \quad (t>0), \qquad \Gamma(x,t) = 0 \quad (t<0)$$

is a fundamental solution of the heat equation, and

$$E(x) = -\frac{1}{(n-2)|S_1||x|^{n-2}} \quad (n \neq 2), \qquad E(x) = \frac{1}{2\pi}\log|x| \quad (n=2)$$

is a fundamental solution of the Laplace equation ($|S_1|$ is the area of the surface of a unit sphere in R^n). The functions $G_{ij}^{(0)}$ constitute a fundamental solution of the problem (2.1) which was contructed in [2,15] for the case $n = 3$ and called "the second fundamental solution". In the n-dimensional case, this solution has the same form and the main properties as for $n = 3$. It was shown in [15] (also for $n = 3$) that

$$|G_{ij}^{(0)}(x,t)| \leq \frac{cx_n^\lambda}{t^{(1+\lambda)/2}(|x|^2+t)^{n/2}}, \quad \lambda \in [0,1], \tag{2.4}$$

$$|D_{x'}^l D_t^m G_{ij}^{(0)}(x,t)| \leq \frac{cx_n^\lambda}{t^{m+(1+\lambda)/2}(|x|^2+t)^{(n+l)/2}}, \tag{2.5}$$

$$|D_{x'}^l D_{x_n}^k D_t^m G_{ij}^{(0)}(x,t)| \leq \frac{c}{t^{m+(1+\lambda)/2}(|x|^2+t)^{(n+l)/2}(x_n^2+t)^{k/2}}. \tag{2.6}$$

It follows that if $|x-z| \leq \theta|x|$, $\theta \in (0,1)$, then

$$|G_{ij}^{(0)}(x,t) - G_{ij}^{(0)}(z,t)| \leq \frac{c|x-z|^\lambda}{t^{(1+\lambda)/2}(|x|^2+t)^{n/2}} \tag{2.7}$$

The above estimates and the equations

$$\int_0^\infty \int_{R^{n-1}} G_{ij}^{(0)}(x,t)dx'dt = \delta_{ij}, \quad \forall x_n > 0, \quad \lambda \in (0,1/2),$$

guarantee that the vector field (2.2) is bounded and continuous in R_+^n, and its limiting value on $R^{n-1} = \{x_n = 0\}$ equals $\vec{a}(x',t)$, for arbitrary bounded and continuous $a_j(x',t)$, moreover, the derivatives of $\vec{v}$ with respect to x_j are continuous for x_n strictly positive. As for the pressure and the time derivative $D_t\vec{v}$, the analysis shows that the expression (2.3) makes sense, if a_j are Hölder continuous with respect to t with the exponent $\beta > 1/2$ (see [15], §7,8). Otherwise $\vec{v}$ satisfies the Stokes equations in a weak sense.

We shall often write the potential (2.2) in the form

$$\vec{v}(x,t) = \int_0^t \int_{R^{n-1}} \mathcal{G}^{(0)}(x'-y', x_n, t-\tau)\vec{a}(y',\tau)dy'd\tau$$

where $\mathcal{G}^{(0)}$ is a tensor which can be defined by

$$\mathcal{G}^{(0)}(x,t) = -2\frac{\partial\Gamma(x,t)}{\partial x_n}(I - \vec{e}_n \otimes \vec{e}_n) + 4\Big((\nabla - \vec{e}_n\frac{\partial}{\partial x_n}) \otimes \vec{Q}^{(0)}(x,t)\Big)^T,$$

$\vec{e}_n = (0,...,0,1)$, $\vec{Q}^{(0)} = (Q_1^{(0)},...,Q_n^{(0)})$. It is easily seen that $\mathcal{G}^{(0)}\vec{a} = \sum_{j=1}^{n-1} G_{ij}^{(0)} a_j$ for arbitrary $\vec{a}$ with $a_n = 0$.

Now, let us consider the problem

$$\vec{v}_t - \Delta\vec{v} + \nabla p = 0, \qquad \nabla\cdot\vec{v} = 0, \qquad x \in R_+,\ t > 0,$$

$$\vec{v}|_{t=0} = 0, \qquad \vec{v}|_R = \vec{a}(\xi,t) \tag{2.8}$$

where R is the plane $\vec{n}\cdot(\vec{x} - \vec{\xi}_0) = 0$, $\vec{n} = (n_1,...,n_n)$ is a unit normal to R, $\xi_0 \in R$ and $R_+ = \{\vec{n}\cdot(\vec{x} - \vec{\xi}_0) > 0\}$. Let $\mathcal{C} = (c_{kj})_{k,j=1,...,n}$ be a constant orthogonal matrix such that

$$c_{nj} = n_j.$$

The transformation

$$z = \mathcal{C}(x - \xi_0), \qquad \vec{u} = \mathcal{C}\vec{v}$$

reduces the problem (2.8) to (2.1), therefore

$$v_i(x,t) = \sum_{k,m=1}^{n} c_{ki} \sum_{j=1}^{n-1} \int_0^t \int_{R^{n-1}} G_{kj}^{(0)}(z - \bar{\zeta}, t-\tau) c_{jm} a_m(\bar{\zeta},\tau) d\zeta' d\tau, \tag{2.9}$$

where $\bar{\zeta} = (\zeta', 0)$, or, which is the same,

$$\vec{v}(x,t) = \int_0^t \int_R \mathcal{G}(x,\xi,t-\tau)\vec{a}(\xi,\tau)dS_\xi d\tau$$

with

$$\mathcal{G}(x,\xi,t) = -2\frac{\partial\Gamma(x-\xi,t)}{\partial n}(I - \vec{n}\otimes\vec{n})$$

$$+4\Big((\nabla - \vec{n}\frac{\partial}{\partial n}) \otimes \int_0^{\vec{n}\cdot(x-\xi)} dy_n \int_{R^{n-1}} \frac{\partial\Gamma(y,t)}{\partial y_n} \nabla E(\mathcal{C}(x-\xi) - y)dy'\Big)^T, \tag{2.10}$$

$\frac{\partial}{\partial n} = \vec{n}\cdot\nabla$. Making the change of variables of integration

$$\eta = \mathcal{C}^{-1}y + \xi_0$$

in the last integral, we obtain

$$\mathcal{G}(x,\xi,t) = -2\frac{\partial\Gamma(x-\xi,t)}{\partial n}(I - \vec{n}\otimes\vec{n})$$

$$+4\Big((\nabla-\vec{n}\frac{\partial}{\partial n})\otimes\int_{\Pi(x)}\frac{\partial\Gamma(\eta-\xi_0,t)}{\partial n}\nabla E(x-\xi-\eta+\xi_0)d\eta\Big)^T \tag{2.11}$$

where $\Pi(x)$ is a layer between R and the parallel plane passing through the point x.

Now, let Ω be a convex domain in R^n with a smooth boundary $S\in C^{1+\gamma}$, $\gamma\in(0,1]$, and let $\vec{n}(\xi)$ be a unit interior normal to S at the point ξ. Consider the problem

$$\vec{v}_t-\Delta\vec{v}+\nabla p=0,\qquad \nabla\cdot\vec{v}=0,\qquad x\in\Omega,\ t\in(0,T),$$

$$\vec{v}|_{t=0}=0,\qquad \vec{v}|_S=\vec{a}(\xi,t) \tag{2.12}$$

whose solution we are going to represent in the form of a hydrodynamic potential. By this we mean the expression

$$\vec{u}(x,t)=\int_0^t\int_S\mathcal{G}(x,\xi,t-\tau)\vec{\phi}(\xi,\tau)dS_\xi \tag{2.13}$$

where $\vec{\phi}$ is a vector field satisfying the condition

$$\vec{\phi}(\xi,t)\cdot\vec{n}(\xi)=0 \tag{2.14}$$

and $\mathcal{G}(x,\xi,t)$ is a tensor of the second fundamental solution corresponding to the tangent plane $R(\xi)$ to S at the point ξ. It can be written in the forms (2.10), (2.11) with $\vec{n}=\vec{n}(\xi)$ and $\xi_0=\xi$, however it should be kept in mind that a smooth orthogonal matrix $C(\xi)$ possessing the property

$$c_{nj}(\xi)=n_j(\xi) \tag{2.15}$$

not always exists on the whole surface S.

The condition of convexity of Ω is necessary because the function $Q_n(z,t)$ has a jump of the first derivative with respect to z_n on the whole plane $z_n=0$.

Together with the function

$$q(x,t)=-4\Big(\frac{\partial}{\partial t}-\Delta\Big)\int_0^t\int_S\vec{Q}(x,\xi,t-\tau)\cdot\vec{\phi}(\xi,\tau)dS,$$

$$\vec{Q}(x,\xi,t)=\int_{\Pi(x,\xi)}\frac{\partial\Gamma(\eta-\xi,t)}{\partial n(\xi)}(\nabla-\vec{n}(\xi)(\vec{n}(\xi)\cdot\nabla))E(x-\eta)d\eta$$

which is well defined, when $\vec{\phi}(\xi,t)$ is Hölder continuous with the exponent $\beta>1/2$ with respect to t, the vector field $\vec{u}(x,t)$ satisfies the Stokes equations, moreover, $\vec{u}(x,0)=0$. Let us compute the limiting value of $\vec{u}(x,t)$ at an arbitrary point $x_0\in S$ assuming that x tends to x_0 remaining inside a certain finite circular cone $\mathcal{K}$ with the vertex x_0 and with the symmetry axis directed along $\vec{n}(x_0)$. We pass to the local Cartesian coordinate system centered at x_0 (we can simply assume that $x_0=0$ and that the x_n-axis is directed along $\vec{n}(x_0)$, then $\mathcal{K}=\{|x'|\le c_0x_n\}$). Let $x_n=F(x')$, $x'=(x_1,...,x_{n-1})$, $|x'|\le d$ be the equation of S in the d-neighbourhood of

the point $x_0 = 0$ and let $\zeta(x)$ be a smooth cut-off function equal to one for $|x| < d/2$ and to zero for $|x| > d$. We have

$$\vec{u}(x,t) = \int_0^t \int_S \mathcal{G}(x,\xi,t-\tau)\vec{\phi}(\xi,\tau)\zeta(\xi)dS_\xi d\tau$$

$$+\int_0^t \int_S \mathcal{G}(x,\xi,t-\tau)\vec{\phi}(\xi,\tau)(1-\zeta(\xi))dS_\xi d\tau \equiv \vec{u}_1(x,t)+\vec{u}_2(x,t).$$

It is clear that

$$\lim_{x\to x_0} \vec{u}_2(x,t) = \vec{u}_2(x_0,t)$$

and

$$\vec{u}_1(x,t) = \int_0^t \int_{|\xi'|\le d} \mathcal{G}(x,\xi,t-\tau)\vec{\phi}(\xi,\tau)\zeta(\xi)\sqrt{1+|\nabla F(\xi')|^2}d\xi' d\tau$$

$$= \int_0^t \int_{|\xi'|\le d} \mathcal{C}^T(\xi)(\mathcal{G}^{(0)}(\mathcal{C}(x-\xi),t-\tau)$$

$$-\mathcal{G}^{(0)}(x-\bar{\xi},t-\tau))\mathcal{C}(\xi)\vec{\phi}(\xi,\tau)\zeta(\xi)\sqrt{1+|\nabla F(\xi')|^2}d\xi' d\tau$$

$$+\int_0^t \int_{R(0)} \mathcal{C}^T(\xi)\mathcal{G}^{(0)}(x'-\bar{\xi},t-\tau)\mathcal{C}(\xi)\vec{\phi}(\xi,\tau)\zeta(\xi)\sqrt{1+|\nabla F(\xi')|^2}d\xi' d\tau \qquad (2.16)$$

where $\xi = (\xi', F(\xi'))$, $\bar{\xi} = (\xi', 0)$ and $\mathcal{C}(\xi)$ is a differentiable orthogonal matrix defined in the d-neighbourhood of the origin and possessing the properties (2.15) and $\mathcal{C}(0) = I$. Such a matrix can be easily constructed.

The difference $\mathcal{G}^{(0)}(\mathcal{C}(x-\xi),t-\tau) - \mathcal{G}^{(0)}(x-\bar{\xi},t-\tau)$ can be estimated with the help of (2.7). We claim that if $x \in \mathcal{K}$, $|\xi'| \le d$ and d is small enough, then

$$|\mathcal{C}(x-\xi)-(x-\bar{\xi})| \le \theta|x-\bar{\xi}|, \qquad \theta\in(0,1). \qquad (2.17)$$

Indeed, since

$$|\xi-\bar{\xi}| = |F(\xi')| \le c|\xi'|^{1+\gamma},$$
$$|\xi'| \le |x'-\xi'| + c_0x_n \le (1+c_0)|x-\bar{\xi}|,$$

we have

$$|\mathcal{C}(x-\xi)-(x-\bar{\xi})| \le |(\mathcal{C}-I)(x-\bar{\xi})| + |\mathcal{C}(\xi-\bar{\xi})|$$
$$\le c(||\xi||x-\xi| + |F(\xi')|) \le c|\xi'|^\gamma|x-\bar{\xi}|$$

which implies (2.17), if d is sufficiently small. Hence, due to (2.7),

$$|G_{ij}^{(0)}(\mathcal{C}(x-\xi),t-\tau) - G_{ij}^{(0)}(x-\bar{\xi},t-\tau)| \le \frac{c|\xi'|^\lambda|x-\xi|^\lambda}{t^{(1+\lambda)/2}(|x-\bar{\xi}|^2+t)^{n/2}}$$

$$\le \frac{c}{t^{(1+\lambda)/2}(|x-\bar{\xi}|^2+t)^{(n-\lambda-\gamma\lambda)/2}}. \qquad (2.18)$$

This estimate, together with (2.4), (2.7), makes it possible to pass to the limit in (2.16) which leads to

$$\lim_{x\to 0}\vec{u}_1(x,t)=\vec{\phi}(0)+\int_0^t\int_S \mathcal{G}(0,\xi,t-\tau)\vec{\phi}(\xi,\tau)\zeta(\xi)dS_\xi d\tau$$

because $\mathcal{G}^{(0)}(-\bar{\xi},t-\tau)=0$. Hence,

$$\lim_{x\to x_0}\vec{u}(x,t)=\vec{\phi}(x_0)+\int_0^t\int_S \mathcal{G}(x_0,\xi,t-\tau)\vec{\phi}(\xi,\tau)dS_\xi d\tau, \qquad \forall x_0\in S. \tag{2.19}$$

Inequality (2.18) with $x=x_0$ shows that the kernel $\mathcal{G}(x_0,\xi,t-\tau)$ is weakly singular, so integral in the right hand side (the direct value of the potential (2.13) on S) is convergent. We denote it by $\mathcal{G}[\vec{\Phi}]$. We need some estimates if this integral.

Lemma 1. *If $S\in C^{1+\gamma}$, $\gamma\in(0,1]$) then the vector field $\vec{u}(x,t)=\mathcal{G}[\vec{\Phi}]$ satisfies the inequalities*

$$\sup_{x\in S}|\vec{u}(x,t)|\le c\int_0^t \sup_{\xi\in S}|\vec{\phi}(\xi,t)|\frac{d\tau}{(t-\tau)^{1-\gamma\lambda/2}}, \qquad \lambda\in(0,1/(1+\gamma)), \tag{2.20}$$

$$|\vec{u}(x,t)|_{C^\alpha(S)}\le c(T)\int_0^t \sup_{\xi\in S}|\vec{\phi}(\xi,t)|\frac{d\tau}{(t-\tau)^{1-\beta/2}}, \tag{2.21}$$

where $\alpha\in(0,\gamma/(1+\gamma))$, $\beta\in(0,\gamma/(1+\gamma)-\alpha)$, $t<T$ and

$$|\vec{u}(x,t)|_{C^\alpha(S)}=\sup_S|\vec{u}(x,t)|+\sup_{x,\xi\in S}|x-\xi|^{-\alpha}|\vec{u}(x,t)-\vec{u}(\xi,t)|$$

is the Hölder norm on S.

Proof. Let us show that for arbitrary $x,\xi,z\in S$

$$|\mathcal{G}(x,\xi,t)|\le\frac{c}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{n/2-\lambda(1+\gamma)/2}}, \tag{2.22}$$

$$|\mathcal{G}(x,\xi,t)-\mathcal{G}(z,\xi,t)|\le\frac{c|x-z|^\alpha}{t^{\alpha+(1+\lambda)(1-\alpha)/2}(|x-\xi|^2+t)^{n/2-\lambda(1-\alpha)(1+\gamma)/2}}, \tag{2.23}$$

if $|x-z|\le\theta|x-\xi|$, $\theta\in(0,1)$.

These estimates follow from (2.4),(2.7). Assume that the point x in (2.4) belongs to the surface $x_n=F(x')$ with $|F(x')|\le c|x'|^{1+\gamma}$. Then

$$|G_{ij}(x,t)|\le\frac{c}{t^{(1+\lambda)/2}(|x|^2+t)^{(n-\lambda-\gamma\lambda)/2}},$$

and if x and z are on the same surface and $|x-z|\le\theta|x|$, then

$$|G_{ij}(x,t)-G_{ij}(z,t)|$$

$$\le c\Big(\frac{|x-z|}{t(|x|^2+t)^{n/2}}\Big)^\alpha\Big(\frac{x_n^\lambda}{t^{(1+\lambda)/2}(|x|^2+t)^{n/2}}+\frac{z_n^\lambda}{t^{(1+\lambda)/2}(|z|^2+t)^{n/2}}\Big)^{1-\alpha}$$

$$\leq \frac{c|x-z|^{\alpha}}{t^{\alpha+(1+\lambda)(1-\alpha)/2}(|x|^2+t)^{n/2-\lambda(1-\alpha)}}.$$

Making the coordinate transformation $z = \mathcal{C}(\xi)(x-\xi)$, we easily deduce (2.22) and (2.23) from the last two inequalities.

Further we have

$$|\vec{u}(x,t)| \leq c\int_0^t \sup_S |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{(1+\lambda)/2}} \int_S \frac{dS}{(|x-\xi|^2+t-\tau)^{n/2-\lambda(1+\gamma)/2}}$$

$$\leq c\int_0^t \sup_S |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{1-\lambda\gamma/2}}, \qquad \lambda(1+\gamma)<1,$$

$$|\vec{u}(x,t)-\vec{u}(z,t)| \leq \int_0^t \int_{s(x)} |\mathcal{G}(x,\xi,t-\tau)-\mathcal{G}(z,\xi,t-\tau)||\vec{\phi}(\xi,\tau)|dS_\xi d\tau$$

$$+\int_0^t \int_{S\setminus s(x)} (|\mathcal{G}(x,\xi,t-\tau)|+|\mathcal{G}(z,\xi,t-\tau)|)|\vec{\phi}(\xi,\tau)|dS_\xi d\tau$$

where $s(x) = \{\xi \in S: \ |\xi - x| \geq 2|x-z|\}$. We estimate the first term in the right hand side using the inequality (2.23) with $\alpha < \lambda(1-\alpha)\gamma < \gamma/(1+\gamma)$ and take into account that $|x-z| > |x-\xi|/2$ in the second term. As a result, we obtain

$$|\vec{u}(x,t)-\vec{u}(z,t)| \leq c|x-z|^{\alpha}\int_0^t \sup_S |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{1-(\alpha-\lambda\gamma(1-\alpha))/2}}$$

$$+c(T)|x-z|^{\alpha}\int_0^t \sup_S |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{1-(\lambda-\alpha)/2}} \leq c|x-z|^{\alpha}\int_0^t \sup_S |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{1-\beta/2}}$$

with $\beta = \lambda\gamma(1-\alpha)-\alpha \in (0,\gamma/(1+\gamma))$. The proposition is proved.

Let us turn to the problem (2.12). Following [13,5], we look for the solution of this problem in the form

$$\vec{v}(x,t) = \int_0^t \int_S \mathcal{G}(x,\xi,t-\tau)\vec{\phi}(\xi,\tau)dSd\tau + 2\nabla\int_S E(x-y)\psi(\xi,t)dS, \tag{2.24}$$

$$p(x,t) = -4(\frac{\partial}{\partial t}-\Delta)\int_0^t \int_S \vec{Q}(x,\xi,t-\tau)\cdot\vec{\phi}(\xi,\tau)dSd\tau - 2\frac{\partial}{\partial t}\int_S E(x-\xi)\psi(\xi,t)dS$$

with $\vec{\phi}(\xi,t)$, $\psi(\xi,t)$ satisfying the conditions (2.14) and

$$\int_S \psi(\xi,t)dS = 0. \tag{2.25}$$

The boundary conditions lead to the following system of the Volterra-Fredholm type:

$$\vec{\phi}(\xi,t) + \mathcal{G}_{tan}[\vec{\phi}] + V_{\tan}[\psi] = \vec{a}(\xi,t),$$

$$\psi(\xi,t) + V_n[\psi] + \mathcal{G}_n[\vec{\phi}] = 0 \qquad \xi \in S. \tag{2.26}$$

Here $\mathcal{G}_{tan}[\vec{\phi}]$ and $\mathcal{G}_n[\vec{\phi}]$ are the tangential and the normal component of $\mathcal{G}[\vec{\phi}]$, i.e.,

$$\mathcal{G}_n[\vec{\phi}] = \vec{n}(\xi) \cdot \int_0^t \int_S \mathcal{G}(\xi, \eta, t - \tau) \vec{\phi}(\eta, \tau) dS_\eta d\tau,$$

$$\mathcal{G}_{tan}[\vec{\phi}] = \mathcal{G}[\vec{\phi}] - \vec{n}\ \mathcal{G}_n[\vec{\phi}]$$

and similarly

$$V_n[\psi] = 2 \int_S \vec{n}(\xi) \cdot \nabla_\xi E(\xi - \eta)\ \psi(\eta, t) dS,$$

$$V_{tan}[\psi] = 2\nabla \int_S E(x - y)\psi(\xi, t) dS - \vec{n}(\xi) V_n[\psi].$$

Theorem 2. *Let $S \in C^{1+\gamma}$, $\gamma \in (0, 1]$. For arbitrary bounded continuous $\vec{a}(\xi, t)$ satisfying the condition $\vec{a}(\xi, t) \cdot \vec{n}(\xi) = 0$ the system (2.26) has a unique solution such that $\vec{\phi}(\xi, t) \cdot \vec{n}(\xi) = 0$, $\int_S \psi(\xi, t) dS_\xi = 0$ and*

$$\sup_{\Sigma_T} |\vec{\phi}(\xi, t)| + \sup_{t<T} |\psi(\cdot, t)|_{C^\alpha(S)} \le c(T) \sup_{\Sigma_T} |\vec{a}(\xi, t)|, \qquad \Sigma_T = S \times (0, T). \tag{2.27}$$

where $\Sigma_T = S \times (0, T)$ and $\alpha \in (0, \gamma/(1 + \gamma))$.

Proof. As in [13,5], we solve the system (2.26) by successive approximations according to the following scheme: $\vec{\phi}_0 = 0$,

$$\vec{\phi}_{m+1}(\xi, t) + \mathcal{G}_{tan}[\vec{\phi}_{m+1}] + V_{tan}[\psi_{m+1}] = \vec{a}(\xi, t), \tag{2.28}$$

$$\psi_{m+1}(\xi, t) + V_n[\psi_{m+1}] + \mathcal{G}_n[\vec{\phi}_m] = 0 \qquad m = 0, 1, 2, , , \tag{2.29}$$

It is well known from the theory of harmonic functions (see [11]) that for a given $\vec{\phi}_m$ the equation (2.29) has a unique solution ψ_m satisfying the condition (2.25), because the necessary condition $\int_S \mathcal{G}_n[\vec{\phi}_m] dS = 0$ holds due to the solenoidality of hydrodymanic potential and the equation (2.19). After having constructed ψ_{m+1}, we find $\vec{\phi}_{m+1}$ from the Volterra equation (2.28). It is clear that $\vec{\phi}_{m+1} \cdot \vec{n} = 0$. The differences

$$\vec{\omega}_{m+1} = \vec{\phi}_{m+1} - \vec{\phi}_m, \qquad \chi_{m+1} = \psi_{m+1} - \psi_m$$

satisfy the system

$$\vec{\omega}_{m+1}(\xi, t) + \mathcal{G}_{tan}[\vec{\omega}_{m+1}] + V_{\tan}[\chi_{m+1}] = 0,$$

$$\chi_{m+1}(\xi, t) + V_n[\chi_{m+1}] + \mathcal{G}_n[\vec{\omega}_m] = 0 \qquad m = 1, 2, , ,$$

and, as a consequence, the following inequalities:

$$|\chi_{m+1}(\cdot, t)|_{C^\alpha(S)} \le c|\mathcal{G}_n[\vec{\omega}_m]|_{C^\alpha(S)} \le c \int_0^t \sup_S |\vec{\omega}_m(\xi, \tau)| \frac{d\tau}{(t - \tau)^{1-\beta/2}}, \tag{2.30}$$

$$\sup_S |\vec{\omega}_{m+1}(\xi, t)| \le c \sup_S |V_{\tan}[\chi_{m+1}]| \le c|[\chi_{m+1}]|_{C^\alpha(S)} \le c \int_0^t \frac{\sup_S |\vec{\omega}_m(\xi, \tau)| d\tau}{(t - \tau)^{1-\beta/2}},$$

$\beta \in (0, \gamma/(1+\gamma) - \alpha)$ (the estimate of the singular integral $V_{tan}[\psi]$ is given in [11]). Hence,

$$\sup_S |\vec{\omega}_{m+1}(\xi,t)| \leq c^m \int_0^t \frac{dt_1}{(t-t_1)^{1-\beta/2}} \int_0^{t_1} \frac{dt_2}{(t_1-t_2)^{1-\beta/2}} \cdots$$

$$\cdots \int_0^{t_{m-1}} \frac{\sup_S |\vec{\omega}_1(\xi,t_m)| dt_m}{(t_{m-1}-t_m)^{1-\beta/2}} \leq c^m \sup_S \sup_{\tau<t} |\vec{\phi}_1(\xi,\tau)| \frac{t^{m\beta/2}(\Gamma(\beta/2))^{m-1}}{\Gamma(m\beta/2+1)}$$

where $\Gamma(z)$ is the Euler Gamma-function. The series $\sum_{m=1}^{\infty} \sup_S |\vec{\omega}_m|$ is convergent, and

$$\sup_S |\vec{\phi}_{m+1}(\xi,t)| \leq \sum_{k=1}^{m} \sup_S |\vec{\omega}_{k+1}(\xi,t)| + \sup_S |\vec{\phi}_1(\xi,t)| \leq c(T) \sup_S \sup_{\tau<t} |\vec{\phi}_1(\xi,\tau)|. \tag{2.31}$$

The vector field $\vec{\phi}_1$ is determined from the equation (2.28) with $m=0$. Since $\psi_1 = 0$, it satisfies the inequality

$$\sup_S \sup_{\tau<t} |\vec{\phi}_1(\xi,\tau)| \leq c \sup_S \sup_{\tau<t} |\vec{a}(\xi,\tau)|, \tag{2.32}$$

so (2.30)-(2.32) yield the estimate (2.27) for the solution of (2.26) constructed above. The uniqueness follows from the inequality

$$\sup_S |\vec{\phi}(\xi,t)| \leq c \int_0^t \sup_S \sup_{\tau<t} |\vec{\phi}(\xi,\tau)| \frac{d\tau}{(t-\tau)^{1-\beta/2}}$$

for the solution of a homogeneous system (2.26). The theorem is proved.

Due to (2.4), the tensor $\mathcal{G}(x,\xi,t)$ satisfies the estimate

$$|\mathcal{G}(x,\xi,t)| \leq \frac{\operatorname{dist}^{\lambda}(x,S)}{t^{(1+\lambda)/2}(|x-\xi|^2+t)^{n/2}}$$

for arbitrary $x \in \omega$, so for the vector field (2.24) we have

$$|\vec{v}(x,t)| \leq c\Big(\sup_{\tau<t} \sup_S |\vec{\phi}(\xi,\tau)| + \sup_{\tau<t} |\psi(\cdot,t)|_{C^{\alpha}(S)}\Big) \leq c \sup_{\tau<t} \sup_S |\vec{a}(\xi,\tau)|, \tag{2.33}$$

i.e., the inequality (1.2) is proved for the solution of the problem (2.12).

The representation formulas (2.24) hold also for the solution of the problem (2.12) with $\vec{a} \cdot \vec{n} \neq 0$. In this case the system of integral equations (2.26) takes the form

$$\vec{\phi}(\xi,t) + \mathcal{G}_{tan}[\vec{\phi}] + V_{\tan}[\psi] = \vec{a}_{tan}(\xi,t),$$

$$\psi(\xi,t) + V_n[\psi] + \mathcal{G}_n[\vec{\phi}] = \vec{a}(\xi,t) \cdot \vec{n}(\xi), \qquad \xi \in S.$$

$\vec{a}_{tan} = \vec{a} - \vec{n}(\vec{a} \cdot \vec{n})$. Instead of (2.27) and (2.33), we have

$$\sup_{\Sigma_T} |\vec{\phi}(\xi,t)| + \sup_{t<T} |\psi(\cdot,t)|_{C^{\alpha}(S)} \leq c(T)\Big(\sup_{\Sigma_T} |\vec{a}_{tan}(\xi,t)| + \sup_{t<T} |\vec{a} \cdot \vec{n}|_{C^{\alpha}(S)}\Big),$$

$$\sup_{\Omega}\sup_{t<T}|\vec{v}(x,t)| \le c\Big(\sup_{S}\sup_{t<T}|\vec{a}_{tan}(\xi,t)| + \sup_{t<T}|\vec{a}(\cdot,t)\cdot\vec{n}|_{C^{\alpha}(S)}\Big).$$

The estimate of the type (2.33) is hardly possible in this case which can be seen also from the explicit formula for the solution of the problem (2.1) (see [15]).

3. Proof of Theorem 1.

In this section we consider the problem (1.1). We reduce it to (2.12) by construction of some auxiliary vector fields. Let $\vec{v}_0^*(x)$ be an extension of $\vec{v}_0(x)$ which is continuous and satisfies the inequality

$$\sup_{R^n}|\vec{v}_0^*(x)| \le c\sup_{\Omega}|\vec{v}_0(x)|,$$

(if we want to construct a regular solution of (1.1), then the extension $\vec{v}_0^*$ should also be regular but at the moment the continuity is enough). Further, let

$$\vec{V}_0(x,t) = \int_{R^n}\Gamma(x-y,t)\vec{v}_0^*(y)dy.$$

This vector field is continuous in x and t, satisfies the heat equation, initial condition $\vec{V}_0(x,0) = \vec{v}_0^*(x)$ and the inequalities

$$\sup_{R^n}|\vec{V}_0(x,t)| \le c\sup_{\Omega}|\vec{v}_0(x)|,$$

$$\sup_{R^n}|D_t^m D_x^j\vec{V}_0(x,t)| \le ct^{-m-|j|/2}\sup_{\Omega}|\vec{v}_0(x)|. \tag{3.1}$$

Further, we define $\vec{v}_1(x,t)$ as a solution to the problem

$$\vec{v}_{1t} - \Delta\vec{v}_1 = 0, \quad x\in\Omega, \quad t\in(0,T), \qquad \vec{v}_1(x,0) = \vec{v}_0(x),$$

$$\vec{v}_1(x,t)|_{x\in S} = \vec{V}_0(x,t) - \vec{n}(x)(\vec{n}\cdot\vec{V}_0(x,t))|_{x\in S} \equiv \vec{a}_1(x,t),$$

and finally we set $\vec{v}_2(x,t) = \nabla\Phi(x,t)$ where Φ is a solution of the Neumann problem

$$\Delta\Phi(x,t) = -\nabla\cdot\vec{v}_1(x,t), \qquad x\in\Omega, \qquad \frac{\partial\Phi}{\partial n}\Big|_S = 0,$$

Then the problem (1.1) is transformed into

$$\vec{u}_t - \Delta\vec{u} + \nabla q = 0, \quad \nabla\cdot\vec{u}(x,t) = 0, \qquad x\in\Omega,\ t\in(0,T),$$

$$\vec{u}(x,0) = 0, \qquad \vec{u}(x,t)|_{x\in S} = \vec{b}(x,t) \tag{3.2}$$

where $\vec{u} = \vec{v} - \vec{v}_1 - \vec{v}_2$, $q = p + \Phi_t$, and

$$\vec{b}(x,t) = \vec{a}(x,t) - \vec{a}_1(x,t) - \vec{v}_2(x,t)|_{x\in S}$$

is a vector field satisfying the conditions $\vec{b}(x,t)\cdot\vec{n}(x) = 0$ and $\vec{b}(x,0) = 0$.

It is necessary to estimate $\vec{v}_1$ and $\vec{v}_2$.

Lemma 2. *There hold the inequalities*

$$\sup_{\Omega} |\vec{v}_1(x,t)| \leq c(\sup_{\Omega} |\vec{v}_0(x)| + \sup_{S} \sup_{\tau<t} |\vec{a}_1(x,\tau)|) \leq c \sup_{\Omega} |\vec{v}_0(x)|, \tag{3.3}$$

$$|\vec{v}_1(\cdot,t)|_{C^\alpha(\Omega)} \leq ct^{-\alpha/2} \sup_{\Omega} |\vec{v}_0(x)|, \tag{3.4}$$

$$|\nabla \vec{v}_1(\cdot,t)|_{C^\alpha(\Omega)} \leq ct^{-(1+\alpha)/2} \sup_{\Omega} |\vec{v}_0(x)|, \tag{3.5}$$

where

$$|u|_{C^\alpha(\Omega)} = \sup_{\Omega} |u(x)| + [u]_\Omega^{(\alpha)}, \qquad [u]_\Omega^{(\alpha)} = \sup_{x,y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Proof. The inequality (3.3) is evident, and (3.4), (3.5) follow from local estimates of solutions of parabolic initial-boundary problems in anisotropic Hölder norms $C^{2+\alpha,1+\alpha/2}(\Omega \times (t_1,t_2)$ (see [12]):

$$|\vec{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_t')} \leq c\Big(|\vec{V}_0|_{C^{2+\alpha,1+\alpha/2}(\Sigma_t'')} + t^{-1-\alpha/2} \sup_{Q_t''} |\vec{v}_1(x,\tau)|\Big)$$

where $Q_t' = \Omega \times (3t/4,t)$, $Q_t'' = \Omega \times (t/2,t)$, $\Sigma_t'' = S \times (t/2,t)$ (here the condition $S \in C^{2+\alpha}$ is used). Making use of the interpolation inequality

$$|D_x^j \vec{v}_1(\cdot,t)|_{C^\alpha(\Omega)} \leq c\Big(t^{(2-|j|)/2} |\vec{v}_1(\cdot,t)|_{C^{2+\alpha}(\Omega)} + t^{-(|j|+\alpha)/2} \sup_{\Omega} |\vec{v}_1(x,t)|\Big) \quad |j| = 0,1,$$

and taking account of (3.3) and (3.1) we obtain (3.4),(3.5). The lemma is proved.

Let us consider the function

$$\Phi(x,t) = \Phi_1(x,t) + \Phi_2(x,t)$$

where

$$\Phi_1(x,t) = -\int_\Omega \nabla E(x-y) \cdot \vec{v}_1(y,t)dy = -\int_\Omega E(x-y)\nabla \cdot \vec{v}_1(y,t)dy$$

and Φ_2 is a solution to the problem

$$\Delta\Phi_2 = 0, \qquad \frac{\partial \Phi_2}{\partial n}\Big|_S = -\frac{\partial \Phi_1}{\partial n}\Big|_S.$$

Lemma 3. *The function Φ_1 satisfies the inequalities*

$$\sup_{\Omega} |\nabla\Phi_1(x,t)| \leq c \sup_{\Omega} |\vec{v}_0(x)|, \tag{3.6}$$

$$[\nabla\Phi_1(\cdot,t)]_\Omega^{(\alpha)} \leq ct^{-\alpha/2} \sup_{\Omega} |\vec{v}_0(x)|. \tag{3.7}$$

Proof. We fix $x_0 \in \Omega$ and we write Φ_1 as a sum $\Phi_1 = \Phi_1' + \Phi_1''$ where

$$\Phi_1'(x,t) = -\int_\Omega \nabla_x E(x-y) \cdot \zeta(y,t)\vec{v}_1(y,t)dy = -\int_\Omega E(x-y)\nabla \cdot \zeta(y,t)\vec{v}_1(y,t)dy,$$

$$\Phi_1''(x,t) = -\int_\Omega \nabla_x E(x-y)\cdot(1-\zeta(y,t))\vec{v}_1(y,t)dy,$$

and $\zeta(y,t) = \zeta_0((y-x_0)/\sqrt{t})$, $\zeta_0(z)$ being a smooth cut-off function equal to one for $|z| \le 1/2$ and to zero for $|z| \ge 1$. For x close to x_0 we have

$$\frac{\partial \Phi_1'(x,t)}{\partial x_m} = -\int_\Omega E_{x_m}(x-y)\nabla\cdot\zeta(y,t)(\vec{v}_1(y,t)-\vec{v}_1(x,t))dy$$

$$-\vec{v}_1(x,t)\cdot\int_\Omega E_{x_m}(x-y)\nabla\zeta(y,t)dy = -\int_\Omega \nabla_x E_{x_m}(x-y)\cdot\zeta(y,t)(\vec{v}_1(y,t)-\vec{v}(x,t))dy$$

$$-\int_S E_{x_m}(x-y)\zeta(y,t)(\vec{v}_1(y,t)-\vec{v}(x,t))\cdot\vec{n}(y)dS_y - \vec{v}_1(x,t)\cdot\int_\Omega E_{x_m}(x-y)\nabla\zeta(y,t)dy,$$

hence, due to (3.3), (3.4),

$$\left|\frac{\partial\Phi_1'(x,t)}{\partial x_m}\Big|_{x=x_0}\right| \le c[\vec{v}_1(\cdot,t)]_\Omega^{(\alpha)}\Big(\int_{|y-x_0|\le\sqrt{t}}\frac{dy}{|y-x_0|^{n-\alpha}} + \int_{\sigma(t)}\frac{dS_y}{|y-x_0|^{n-1-\alpha}}\Big)$$

$$+c|\vec{v}_1(x_0,t)| \le c\sup_\Omega|\vec{v}_0(x)|$$

where $\sigma(t) = \{y \in S : |y-x_0| \le \sqrt{t}\}$.

Since $\vec{v}_1$ satisfies the heat equation, the function Φ_1'' may be written in the form

$$\Phi_1''(x,t) = -\int_\Omega \nabla_x E(x-y)\cdot(1-\zeta(y,t))\Big(\vec{v}_0(y) + \int_0^t \Delta\vec{v}_1(y,\tau)d\tau\Big)dy,$$

hence,

$$\frac{\partial\Phi_1''(x,t)}{\partial x_m} = \int_\Omega E_{x_m}(x-y)\nabla\zeta(y,t)\cdot\vec{v}_0(y)dy$$

$$+\sum_{k=1}^n\int_0^t d\tau\int_\Omega \frac{\partial^2\nabla_x E(x-y)(1-\zeta(y,t))}{\partial y_k\partial x_m}\cdot\Big(\frac{\partial\vec{v}_1(y,\tau)}{\partial y_k} - \frac{\partial\vec{v}_1(x,\tau)}{\partial x_k}\Big)dy$$

$$-\sum_{k=1}^n\int_0^t d\tau\int_S \nabla_x E_{x_m}(x-y)\cdot(1-\zeta(y,t))\Big(\frac{\partial\vec{v}_1(y,\tau)}{\partial y_k} - \frac{\partial\vec{v}_1(x,\tau)}{\partial x_k}\Big)n_k(y)dS_y,$$

and

$$\left|\frac{\partial\Phi_1''(x,t)}{\partial x_m}\Big|_{x=x_0}\right| \le c\Big(\sup_\Omega|\vec{v}_0(y)|$$

$$+\int_0^t[\nabla\vec{v}_1(\cdot,\tau)]_\Omega^{(\alpha)}d\tau\Big(\int_{|y-x_0|\le\sqrt{t}/2}\frac{dy}{|y-x_0|^{n+1-\alpha}} + \int_{S\setminus\sigma(t/4)}\frac{dS_y}{|y-x_0|^{n-\alpha}}\Big)\Big)$$

$$\le c\Big(\sup_\Omega|\vec{v}_0(y)| + t^{-(1-\alpha)/2}\int_0^t[\nabla\vec{v}_1(\cdot,\tau)]_\Omega^{(\alpha)}d\tau\Big) \le c\sup_\Omega|\vec{v}_0(y)|,$$

which completes the proof of (3.6).

Estimate (3.7) follows from (3.4). Indeed, making use of classical estimates of the Newtonian potential (see [11]) and of (3.4), we obtain

$$[\nabla\Phi_1(\cdot,t)]_\Omega^{(\alpha)} \le c[\vec{v}_1(\cdot,t)]_\Omega^{(\alpha)} \le ct^{-\alpha/2}\sup_\Omega |\vec{v}_0(x)|.$$

The lemma is proved.

Lemma 4. *The function* Φ_2 *satisfies the inequality* (3.6).

Proof. The function $\Phi_2(x,t)$ is representable in the form of a simple layer potential

$$\Phi_2(x,t) = 2\int_S E(x-y)\mu(y,t)dS_y$$

where $\mu(y,t)$ is a solution of the integral equation

$$\mu + V_n[\mu] = -\frac{\partial\Phi_1(y,t)}{\partial n}, \qquad y \in S.$$

Since

$$\int_S \frac{\partial\Phi_1(y,t)}{\partial n} dS_y = \int_\Omega \Delta\Phi_1(y,t)dy = -\int_\Omega \nabla\cdot\vec{v}_1(y,t)dy = -\int_S \vec{v}_1(y,t)\cdot\vec{n}(y)dS_y = 0,$$

this equation has a unique solution satisfying the condition (2.25) and the inequalities

$$\sup_S |\mu(y,t)| \le c\sup_S \left|\frac{\partial\Phi_1(y,t)}{\partial n}\right|,$$

$$|V_n[\mu]|_{C^\alpha(S)} \le c\sup_S \left|\frac{\partial\Phi_1(y,t)}{\partial n}\right| \tag{3.8}$$

Hence,

$$\Phi_2(x,t) = -2\int_S E(x-y)\frac{\partial\Phi_1(y,t)}{\partial n}dS_y - 2\int_S E(x-y)V_n[\mu](y,t)dS_y$$
$$\equiv \Phi_3(x,t) + \Phi_4(x,t);$$

using (3.8) and the estimate of the gradient of a simple layer potential given in [11], we obtain

$$\sup_S |\nabla\Phi_4(y,t)| \le c|V_n[\mu]|_{C^\alpha(S)} \le c\sup_S \left|\frac{\partial\Phi_1(y,t)}{\partial n}\right|.$$

It remains to estimate $\nabla\Phi_3$. The function $\psi(x,t) = \frac{\partial\Phi_1(x,t)}{\partial n}|_S$ possesses the properties

$$\sup_S |\psi(x,t)| \le c\sup_\Omega |\vec{v}_0(x)|,$$

$$[\psi(\cdot,t)]_S^{(\alpha)} \le ct^{-\alpha/2}\sup_\Omega |\vec{v}_0(y)|,$$

and, in addition,

$$\psi(x,t) = -\sum_{k,m=1}^{n} n_k \frac{\partial^2}{\partial x_k \partial x_m} \int_\Omega E(x-y) v_{1m}(y,t)dy$$

$$= -\sum_{k,m=1}^{n} \Big(n_k \frac{\partial}{\partial x_m} - n_m \frac{\partial}{\partial x_k}\Big) \frac{\partial}{\partial x_k} \int_\Omega E(x-y) v_{1m}(y,t)dy$$

$$= -\sum_{k,m=1}^{n} \Big(n_k \frac{\partial}{\partial x_m} - n_m \frac{\partial}{\partial x_k}\Big) \frac{\partial}{\partial x_k} \int_0^t d\tau \int_\Omega E(x-y) \Delta v_{1m}(y,\tau)dy. \qquad (3.9)$$

¿From classical estimates of the Newtonian and the simple layer potentials it follows immediately that the functions

$$U_{km}(x,t) = \frac{\partial}{\partial x_k} \int_0^t d\tau \int_\Omega E(x-y)\Delta v_{1m}(y,\tau)dy$$

$$= \sum_{q=1}^{n} \frac{\partial^2}{\partial x_k \partial x_q} \int_0^t d\tau \int_\Omega E(x-y) \frac{\partial v_{1m}(y,t)}{\partial y_q} dy$$

$$+ \frac{\partial}{\partial x_k} \int_0^t d\tau \int_S E(x-y) \frac{\partial v_{1m}(y,\tau)}{\partial n} dS_y$$

satisfy the inequality

$$[U_{km}(\cdot,t)]_S^{(\alpha)} \le c \int_0^t |\nabla \vec{v}_1(\cdot,\tau)|_{C^\alpha(\Omega)} \le ct^{(1-\alpha)/2} \sup_\Omega |\vec{v}_0(x)|.$$

The representation formula (3.9) enables us to "integrate by parts" on S and to estimate $|\nabla\Phi_3|$ with the same kind of arguments as above in Lemma 3. Let $x_0 \in \Omega$ and let $\zeta(y,t)$ be the function defined in Lemma 3. We have

$$\nabla\Phi_3(x,t) = -2\int_S \nabla E(x-y)\zeta(y,t)\psi(y,t)dS_y$$

$$+2\int_S \nabla E(x-y)(1-\zeta(y,t)) \sum_{k,m=1}^{n} \Big(n_k(y)\frac{\partial}{\partial y_m} - n_m(y)\frac{\partial}{\partial y_k}\Big) U_{km}(y,t)dS_y$$

$$= -2\int_S \nabla E(x-y)\zeta(y,t)(\psi(y,t)-\psi(x,t))dS_y - 2\psi(x,t)\int_S \nabla E(x-y)\zeta(y,t)dS_y$$

$$-2\sum_{k,m=1}^{n} \int_S (U_{km}(y,t)-U_{km}(x,t))\Big(n_k\frac{\partial}{\partial y_m} - n_m\frac{\partial}{\partial y_k}\Big)\Big(\nabla E(x-y)(1-\zeta(y,t))\Big)dS_y,$$

and, since the integral $\int_S \nabla E(x-y)\zeta(y,t)dS_y = \int_{\sigma(t)} \nabla E(x-y)\zeta(y,t)dS_y$ is uniformly bounded (see [11]), there holds the estimate

$$|\nabla\Phi_3(x_0,t)| \le c\Big([\psi(\cdot,t)]_S^{(\alpha)} \int_{\sigma(t)} \frac{dS}{|x_0-y|^{n-1-\alpha}} + |\psi(x_0,t)|$$

$$+ \sum_{k,m=1}^{n} [U_{km}(\cdot,t)]_S^{(\alpha)} \int_{S\setminus\sigma(t)} \frac{dS}{|x_0-y|^{n-\alpha}} \Bigg) \le c \sup_{\Omega} |\vec{v}_0(x)|,$$

q.e.d.

It follows that

$$\sup_S \sup_{t<T} |\vec{b}(y,t)| \le c \sup_{\Omega} |\vec{v}_0(x)|;$$

now, making use of inequality (2.33), we estimate the solution of the problem (3.2) and conclude the proof of Theorem 1.

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A note on the blow-up criterion for the inviscid 2-D Boussinesq equations

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Abstract

We show that a smooth solution of the 2-D Boussinesq equations in the whole plane $\boldsymbol{R}^2$ breaks down if and only if a certain norm of the gradient of passive scalar $\nabla\theta$ blows up at the same time. Here the norm is weaker than L^∞-norm and generates a Banach space including singularities of $\log\log 1/|x|$. Roughly speaking, when a smooth solution breaks down, $\nabla\theta$ has stronger singularities than $\log\log 1/|x|$ or has infinite number of singularities.

1 Introduction

In this note, we deal with incompressible inviscid 2-D flows interacting with a passive scalar under the external potential force f, (i.e., rot $f = 0$), which described by the Boussinesq equations:

$$
\text{(B)}\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \theta f, \qquad t\ge 0, \quad x\in \boldsymbol{R}^2, \\ \partial_t\theta + u\cdot\nabla\theta = 0, \qquad t\ge 0, \quad x\in\boldsymbol{R}^2, \\ \quad \operatorname{div} u = 0, \qquad t\ge 0, x\in\boldsymbol{R}^2, \\ \quad u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0, \end{cases}
$$

where $u = (u^1(x,t), u^2(x,t))$, $\theta = \theta(x,t)$ and $p = p(x,t)$ denote the unknown velocity vector field, the unknown scalar (e.g. temperature) and the unknown pressure of the fluid at the point $(x,t)\in\boldsymbol{R}^2\times(0,\infty)$, respectively, while u_0, θ_0 are given initial date.

Let $m\ge 3$ be an integer. It is proved by Chae-Nam [5] that for every $(u_0,\theta_0)\in H^m(\boldsymbol{R}^2)\times H^m(\boldsymbol{R}^2)$with div $a = 0$ and for every $f\in L^\infty_{loc}([0,\infty); W^{m,\infty}(\boldsymbol{R}^2))$, there exist $T>0$ and a unique solution (u,θ) of (E) on $[0,T)$ in the class

$$C([0,T); H^m(\boldsymbol{R}^2)\times H^m(\boldsymbol{R}^2)), \tag{1.1}$$

where T is estimated from below as

$$T \geq C/(\|u_0\|_{H^m} + \|\theta_0\|_{H^m} + \sup_{0<t<T_0} \|f(t)\|_{W^{m,\infty}}) \tag{1.2}$$

for sufficiently large T_0. Moreover they proved that the maximum norm of $\nabla\theta$ controls the breakdown (blow-up) of smooth solution to (B), i.e.,

$$\limsup_{t\nearrow T}(\|u(t)\|_{H^m} + \|\theta(t)\|_{H^m}) = \infty$$

if and only if

$$\int_0^t \|\nabla\theta(\tau)\|_{L^\infty} d\tau \nearrow \infty \quad \text{as} \quad t \nearrow T. \tag{1.3}$$

This shows that the velocity u is irrelevant to the breakdown. See also [4]. We note that in physcs literature [26] Weinan-Shu observed this phenomenon. Recently, Ishimura-Morimoto [12] dealt with the viscous 3-D Boussinesq equations in a periodic domain and obtained the similar result. They proved that the maximum norm of the gradient of vorticity controls the breakdown; the scalar temperature function θ is irrelevant to the breakdown.

Those blow-up criteria are similar to Beale-Kato-Majda's result on 3-D Euler equations in a domain Ω:

$$(\mathrm{E}) \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, & t \geq 0, \quad x \in \Omega, \\ \operatorname{div} u = 0, & t \geq 0, x \in \Omega, \\ u|_{t=0} = u_0, \end{cases}$$

In the celebrated paper [1], Beale-Kato-Majda showed that the maximum norm of the vorticity rot u controls the breakdown of smooth solutions to the Euler equations if Ω is the whole space $\boldsymbol{R}^3$, i.,e.,

$$\limsup_{t\nearrow T} \|u(t)\|_{H^m(\boldsymbol{R}^3)} = \infty \text{ if and only if } \int_0^t \|\mathrm{rot}\ u(\tau)\|_{L^\infty(\boldsymbol{R}^3)} d\tau \nearrow \infty, \quad \text{as} \quad t \nearrow T,$$

(in cace $\Omega = \boldsymbol{R}^N$, see Kato-Ponce [15]). When Ω is a bounded domain, Ferrari [11] and Shirota-Yanagisawa [23] proved the same result of breakdown as in [1]. For $\Omega = \boldsymbol{R}^N$, the result of Beale-Kato-Majda was improved Kozono-Taniuchi [17] so that the blow-up phenomenon is controled by the BMO-norm of vorticity, see also Kozono-Ogawa-Taniuchi [18]. On the other hand, in case Ω is a bounded domain, the blow-up criterion was also improved in [20] and [21]. In particular, it is proved in [21] that

$$\limsup_{t\nearrow T} \|u(t)\|_{H^m(\Omega)} = \infty \text{ if and only if } \int_0^t \|\mathrm{rot}\ u(\tau)\|_{\mathrm{bmo}(\Omega)} d\tau \nearrow \infty \quad \text{as} \quad t \nearrow T.$$

(For the definition of bmo(Ω), see [21].) Moreover in [21] we observed a slightly improved condition for the breakdown. We note that there are orther criteria for

the 3-D Euler equations in $\boldsymbol{R}^3$. Constantin [6] and Constantin-Fefferman-Majda [9] proved that under some additional assumption on the direction of vorticity $\frac{\mathrm{rot}\ u}{|\mathrm{rot}\ u|}$, the $L^1_{unif,loc}$-norm of vorticity controls the breakdown.

Let us return to the 2-D Boussinesq equations. In this note, using the method in [19], [20] and [21], we show that it is also possible to improve Chae-Nam's criterion (1.3). Namely the condition on $\nabla\theta$ for the breakdown is relaxed and the new one permits double-logarithmic singularities. For uniqueness theorem of weak solutions to the Euler equations, Yudovich [27] introduced some classes (Y_ϕ-spaces), which include $\log^+\log^+ 1/|x|$. We introduce a space M_ϕ, which is wider than Y_ϕ and prove that the breakdown of smooth solutions to the 2-D Boussinesq equations is controlled by the M_ϕ-norm of $\nabla\theta$.

It is notable that Vishik [24] recently showed the uniqueness and global existence of solutions to the Euler equations in $\boldsymbol{R}^2$ with unbounded vorticity in a space of Besov type, which wider than Y_ϕ, see also [25].

2 Preliminaries and Main Result

In this section, we introduce some notations and function spaces. Set $H^m_\sigma(\boldsymbol{R}^2) \equiv \{u \in H^m(\boldsymbol{R}^2); \mathrm{div}\ u = 0\}$, where H^m is the usual Sobolev space $W^{m,2}(\boldsymbol{R}^2)$. Let ϕ be a nondecreasing function on $[1,\infty)$. In [27], for Uniqueness theorem of the Euler equations Yudovich introduced the function space Y_ϕ.

Y_ϕ is a set of all functions f in $\bigcap_{1<p<\infty} L^p$ satisfying

$$\|f\|_{L_\phi} \equiv \sup_{p\geq 1} \frac{\|f\|_{L^p}}{\phi(p)} < \infty. \tag{2.1}$$

Similarly to Y_ϕ, we define the following space:

$M_\phi(\boldsymbol{R}^N)$ denotes the set of all functions f in $\bigcap_{p\geq 1} M^p_{loc}(\boldsymbol{R}^N)$ satisfying

$$\|f\|_{M_\phi} \equiv \sup_{p\geq 1} \frac{\|f\|_{M^p_{loc}}}{\phi(p)} < \infty, \tag{2.2}$$

where M^p_{loc} is the local Morrey space with

$$\|f\|_{M^p_{loc}} \equiv \sup_{0<r<1, x\in\boldsymbol{R}^N} \left\{ \int_{|y-x|<r} |f(y)|dy \cdot r^{-N+\frac{N}{p}} \right\}.$$

Obviously M_ϕ is wider than Y_ϕ and we see that if we take $\phi(p) = 1$ for all $p \geq 1$, then $M_\phi \equiv L^\infty$. It is worth to note that

$$\begin{aligned} &\log^+\log^+ \tfrac{1}{|x|} \in Y_\phi \subset M_\phi \text{ for } \phi(p) = \log p, \\ &\log^+ \tfrac{1}{|x|} \cdot \log^+\log^+ \tfrac{1}{|x|} \in Y_\phi \subset M_\phi \text{ for } \phi(p) = p\log p, \end{aligned} \tag{2.3}$$

(see [27]) and

$$\|f\|_{M_\phi} \leq C\|f\|_{L^\infty(\Omega)}, \quad L^\infty \subset M_\phi,$$

where $\log^+ t \equiv \max(0, \log t)$.

The following lemma is obtained by a similar argument found in Engler [10] and Ozawa [22].

Lemma 2.1 (Ogawa-Taniuchi[20],[21]) *Let $s > N/q$, $1 < q < \infty$ and let ϕ be a nondecreasing function on $[1, \infty)$. Then there exists a constant $C = C(\phi, N, q, s) > 0$ such that for all $f \in W^{s,q}(\boldsymbol{R}^N)$*

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{M_\phi} \phi(\log(\|f\|_{W^{s,q}(\boldsymbol{R}^N)} + e))). \tag{2.4}$$

Now we recall the following proposition due to Beale-Kato-Majda [1] and Kato-Ponce [14].

Proposition 2.1 *Let $s > N/p + 1$ and $1 < p < \infty$. Then there exists a constant depending only on N, s and p such that*

$$\|\nabla u\|_{L^\infty(\boldsymbol{R}^N)} \leq C(1 + \|\text{rot } u\|_{L^\infty(\boldsymbol{R}^N)} \log(\|u\|_{W^{s,p}(\boldsymbol{R}^N)} + e)) \tag{2.5}$$

for all $u \in W^{s,p}(\boldsymbol{R}^N)$ with div $u = 0$.

Remarks 1. (i) While Beale-Kato-Majda and Kato-Ponce added the superfluous term $\|\text{rot } u\|_{L^p}$ in the right hand side of their original inequalities in [1] and [14], (2.5) does not need $\|\text{rot } u\|_{L^p}$. For the detail, see [17] and [18], where more general inequalities were proved.

(ii) Ferrari [11] and Shirota-Yangisawa [23] proved (2.5) for a bounded domain $\Omega \subset \boldsymbol{R}^3$ with smooth boundary $\partial\Omega$. Recently, in [21] we slightly improved it as follows:

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C(1 + \|\text{rot } u\|_{bmo(\Omega)} \log(\|u\|_{W^{s,p}(\Omega)} + e))$$

for all $u \in W^{s,p}(\Omega)$ with div $u = 0$ in Ω and $u \cdot n = 0$ on $\partial\Omega$, where $n = n(x) = (n^1(x), n^2(x), n^3(x))$ is the unit outward normal at $x \in \partial\Omega$.

Thanks to Lemma 2.1 and Proposition 2.1, we have the following criterion:

Theorem 1 *Let the initial date $(u_0, \theta_0) \in H^m_\sigma(\boldsymbol{R}^2) \times H^m(\boldsymbol{R}^2)$ for some $m > 2$ and let $f \in L^\infty_{loc}([0, \infty); W^{m,\infty}(\boldsymbol{R}^2))$. Then the solution (u, θ) to (B) in the class $C([0, T); H^m_\sigma \times H^m), (T < \infty)$ satisfies*

$$\limsup_{t \nearrow T} (\|u(t)\|_{H^m} + \|\theta(t)\|_{H^m}) = \infty \tag{2.6}$$

if and only if

$$\int_0^t \|\nabla\theta(\tau)\|_{M_\phi} d\tau \nearrow \infty \quad \text{as} \quad t \nearrow T \quad \text{for } \phi(p) = \log(p + e). \tag{2.7}$$

Remarks 2. (i) Theorem 1 is an improvement of the blow-up criterion of Chae-Nam [5], since $\|f\|_{M_\phi} \le C\|f\|_{L^\infty}$ and M_ϕ is strictly wider than L^∞. Roughly speaking, when a smooth solution breaks down, $\nabla\theta$ has stronger singularities than $\log\log 1/|x|$ or has infinite number of singularities.

(ii) Theorem 1 also holds for every nondecreasing function $\phi(p) \ge 1$ on $[1,\infty)$ with $\int^\infty 1/\phi(e^p)dp = \infty$. For the detail, see [21, Lemma 2.5]. For example, $\phi(p) = \log(p+e)\cdot\log(\log(p+e)+e)$, $\log(p+e)\cdot\log(\log(p+e)+e)\cdot\log(\log(\log(p+e)+e)+e)$, $\cdots$.

(iii) It seems to be true that there holds Theorem 1 for a bounded domain $\Omega \subset \boldsymbol{R}^2$ with smooth boundary $\partial\Omega$ as well as $\boldsymbol{R}^2$.

3 Proof

Let $\|\cdot\|_p$ denote L^p-norm. We prove Theorem 1, according to Chae-Nam [5]. We easily see that (2.7) implies (2.6), since $\|\nabla\theta\|_{M_\phi} \le \|\nabla\theta\|_{L^\infty} \le C\|\theta\|_{H^m}$. Hence it suffices to show that (2.6) imples (2.7). Let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index with $|\alpha| \equiv \alpha_1 + \alpha_2 \le m$, and apply ∂^α to the first equation in (B). Then we have

$$\frac{\partial}{\partial t}\partial^\alpha u + u\cdot\nabla\partial^\alpha u + \nabla\partial^\alpha p = -\partial^\alpha(u\cdot\nabla u) + u\cdot\nabla\partial^\alpha u + \partial^\alpha(\theta f). \tag{3.1}$$

Taking inner product in $L^2(\boldsymbol{R}^2)$ between (3.1) and $\partial^\alpha u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_2^2 \le \|\partial^\alpha(u\cdot\nabla u) - u\cdot\nabla\partial^\alpha u\|_2\|\partial^\alpha u\|_2 + \|\partial^\alpha(\theta f)\|_2\|\partial^\alpha u\|_2, \tag{3.2}$$

since div $u = 0$ yields $(u\cdot\nabla\partial^\alpha u, \partial^\alpha u) = (\nabla\partial^\alpha p, \partial^\alpha u) = 0$. Now we recall the commutator estimate given by Beale-Kato-Majda [1, (13)]:

$$\|\partial^\alpha(fg) - f\partial^\alpha g\|_2 \le C(\|f\|_{H^m}\|g\|_\infty + \|\nabla f\|_\infty\|g\|_{H^{m-1}}), \tag{3.3}$$

see also [15]. The above inequality yields

$$\|\partial^\alpha(u\cdot\nabla u) - u\cdot\nabla\partial^\alpha u\|_2 \le C\|\nabla u\|_\infty\|u\|_{H^m}. \tag{3.4}$$

Hence we have by (3.2) and (3.4)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u\|_2^2 &\le C\|\nabla u\|_\infty\|u\|_{H^m}^2 + \|f\|_{W^{m,\infty}}^2\|\theta\|_{H^m}^2 + \|u\|_{H^m}^2 \\ &\le C(1+\|\nabla u\|_\infty)(\|u\|_{H^m}^2 + \|\theta\|_{H^m}^2)\end{aligned} \tag{3.5}$$

Integrating (3.5) on the time interval $(0,t)$ and summing over α with $0 \le |\alpha| \le m$, we obtain

$$\|u(t)\|_{H^m}^2 \le \|u_0\|_{H^m}^2 + C\int_0^t (1+\|\nabla u(\tau)\|_\infty)(\|u(\tau)\|_{H^m}^2 + \|\theta(\tau)\|_{H^m}^2)d\tau. \tag{3.6}$$

Similarly, from the second equation in (B) we obtain

$$\begin{aligned} &\|\theta(t)\|_{H^m}^2 \\ \le\ &\|\theta_0\|_{H^m}^2 + C\int_0^t (1+\|\nabla u(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)(\|u(\tau)\|_{H^m}^2 + \|\theta(\tau)\|_{H^m}^2)d\tau, \end{aligned} \tag{3.7}$$

where we used $(u\cdot\nabla\partial^\alpha\theta, \partial^\alpha\theta) = 0$ and

$$\|\partial^\alpha(u\cdot\nabla\theta) - u\cdot\nabla\partial^\alpha\theta\|_2 \le C\|u\|_{H^m}\|\nabla\theta\|_\infty + \|\nabla u\|_\infty\|\theta\|_{H^m} \tag{3.8}$$

given by (3.3). Let $\tilde{u}(t) = (u(t),\theta(t))$ and let $\tilde{u}_0 = (u_0,\theta_0)$. The combination of (3.6) with (3.7) implies

$$\|\tilde{u}(t)\|_{H^m}^2 \le \|\tilde{u}_0\|_{H^m}^2 + C\int_0^t (1+\|\nabla u(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)\|\tilde{u}(\tau)\|_{H^m}^2 d\tau. \tag{3.9}$$

Then Gronwall's inequality yields

$$\|\tilde{u}(t)\|_{H^m} \le \|\tilde{u}_0\|_{H^m}\exp C\int_0^t (1+\|\nabla u(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)d\tau, \tag{3.10}$$

which implies

$$\begin{aligned} &\log(\|\tilde{u}(t)\|_{H^m} + e) \\ \le\ &\log(\|\tilde{u}_0\|_{H^m} + e) + C\int_0^t (1+\|\nabla u(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)d\tau. \end{aligned} \tag{3.11}$$

Proposition 2.1 gives

$$\|\nabla u\|_\infty \le C(1+\|\omega\|_\infty\log(\|u\|_{H^m}+e)) \le C(1+\|\omega\|_\infty\log(\|\tilde{u}\|_{H^m}+e)), \tag{3.12}$$

where ω denotes the vorticity, i.e., $\omega = \operatorname{rot} u$. Substituting (3.12) to (3.11), we have

$$\begin{aligned} &\log(\|\tilde{u}(t)\|_{H^m} + e) \\ \le\ &\log(\|\tilde{u}_0\|_{H^m} + e) + C\int_0^t (1+\|\omega(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)\log(\|\tilde{u}(\tau)\|_{H^m}+e)d\tau. \end{aligned} \tag{3.13}$$

Again using Gronwall's inequality, we obtain

$$\begin{aligned} &\log(\|\tilde{u}(t)\|_{H^m} + e) \\ \le\ &\log(\|\tilde{u}_0\|_{H^m} + e)\exp C\int_0^t (1+\|\omega(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)d\tau, \end{aligned} \tag{3.14}$$

which implies

$$\begin{aligned} \log\log(\|\tilde{u}(t)\|_{H^m} + e) \le\ &\log\log(\|\tilde{u}_0\|_{H^m} + e) \\ &+C\int_0^t (1+\|\omega(\tau)\|_\infty + \|\nabla\theta(\tau)\|_\infty)d\tau. \end{aligned} \tag{3.15}$$

Now we use Lagrange variables to estimate $\|\omega\|_\infty$. Let η be the stream lines of the flow u, i.e., the solutions of the ordinary differential equations:

$$\frac{d\eta}{dt}(x,t) = u(\eta(x,t),t), \quad \eta(x,0) = x. \tag{3.16}$$

Application of rot to the first equation in (B) gives

$$\begin{cases} \frac{\partial}{\partial t}\omega + u \cdot \nabla\omega = (\nabla\theta \times f)_3 & \text{in } \boldsymbol{R}^2 \times (0,T) \\ \omega|_{t=0} = \omega_0, \end{cases} \tag{3.17}$$

where $\omega_0 = \operatorname{rot} u_0$ and $(\cdot)_3$ denotes the third component of $(\cdot)$. Then ω has the following representation:

$$\omega(\eta(x,t),t) = \omega_0(x) + \int_0^t (\nabla\theta \times f)_3(\eta(x,s),s)ds. \tag{3.18}$$

Hence there holds

$$\begin{aligned} \|\omega(t)\|_\infty &\le \|\omega_0\|_\infty + \int_0^t \|\nabla\theta(s)\|_\infty \|f(s)\|_\infty ds \\ &\le \|\omega_0\|_\infty + M\int_0^t \|\nabla\theta(s)\|_\infty ds, \end{aligned} \tag{3.19}$$

where $M \equiv \sup_{0\le s\le T}\|f(s)\|_\infty$.

Substituting (3.19) to (3.15), we have

$$\begin{aligned} &\log\log(\|\tilde{u}(t)\|_{H^m} + e) \\ \le\ &\log\log(\|\tilde{u}_0\|_{H^m} + e) \\ &+C\left\{(1+\|\omega_0\|_\infty)t + \int_0^t \|\theta(\tau)\|_\infty d\tau + M\int_0^t\int_0^\tau \|\nabla\theta(s)\|_\infty ds d\tau\right\} \\ \le\ &\log\log(\|\tilde{u}_0\|_{H^m} + e) \\ &+C\left\{(1+\|\omega_0\|_\infty)t + (1+tM)\int_0^t \|\nabla\theta(\tau)\|_\infty d\tau\right\}. \end{aligned} \tag{3.20}$$

Applying Lemma 2.1 with $\phi(p) = \log(p+e)$, we see that

$$\begin{aligned} \|\theta(\tau)\|_\infty &\le C\{1 + \|\nabla\theta(\tau)\|_{M_\phi}\log(\log(\|\theta(\tau)\|_{H^m} + e) + e)\} \\ &\le C\{1 + \|\nabla\theta(\tau)\|_{M_\phi}\log(\log(\|\tilde{u}(\tau)\|_{H^m} + e) + e)\} \\ &\le C\{1 + \|\nabla\theta(\tau)\|_{M_\phi}(1 + \log\log(\|\tilde{u}(\tau)\|_{H^m} + e))\}. \end{aligned} \tag{3.21}$$

Letting $z(t) = \log\log(\|\tilde{u}(t)\|_{H^m} + e)$, from (3.20) and (3.21) we obtain

$$z(t) \le z(0) + L + C(1+T)\int_0^t (1 + \|\nabla\theta(\tau)\|_{M_\phi})(z(\tau)+1)d\tau \text{ for } 0 < t < T, \tag{3.22}$$

where L is a constant depending only on T, $\|\omega_0\|_\infty$, $\sup_{0\le s\le T}\|f(s)\|_{W^{m,\infty}}$ and m. Then Gronwall's inequality yields

$$z(t) \le (z(0)+L+1)\exp\left(C(1+T)\int_0^t (1+\|\nabla\theta(\tau)\|_{M_\phi})d\tau\right) \text{ for } 0<t<T, \quad (3.23)$$

which shows that (2.6) implies (2.7). This completes the proof of Theorem 1.

Remark 3. We notice that (3.20) proves Chae-Nam's blow-up criterion. Since in [5] they used the well-known inequality

$$\|\nabla u\|_\infty \le C(1+\|\text{rot } u\|_\infty(1+\log^+\|u\|_{H^m})+\|\text{rot } u\|_p) \text{ for div } u=0 \quad (0<p<\infty)$$

given in [1] and [14], they needed estimate $\|\text{rot } u\|_p$. On the other hand, using (2.5) instead of the above inequality, we need not to estimate $\|\text{rot } u\|_p$ and hence slightly simplify the proof of Chae-Nam's blow-up criterion.

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Weak Solutions to Viscous Heat-conducting Gas 1D-Equations with discontinuous data: Global Existence, Uniqueness, and Regularity

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1 Introduction

In this paper, we give a review of our recent results on nonhomogeneous initial-boundary value problems to 1D–Navier-Stokes equations for a viscous heat-conducting gas with large discontinuous data. For instance, we consider the initial data such that the specific volume is bounded from above and below and the total initial energy and entropy are finite. The global in time existence of weak solutions [6], their uniqueness and Lipschitz continuous dependence on data [21],[23] as well as an internal and up to the boundary regularity [9] are presented.

Note the related results in [18],[20],[19],[2],[3],[13]–[15]. The most well-known results concerning the equations under consideration belong to A.V.Kazhikhov who studied global regular solutions in the case of the initial and boundary data from the Sobolev space H^1, see Chapter 2 in [11].

Now we introduce notation that will be used throughout the paper. Let $\Omega = (0,X)$ and $Q = Q_T \equiv \Omega \times (0,T)$. We write $Dw = \frac{\partial w}{\partial x}$, $D_t w = \frac{\partial w}{\partial t}$, and $(I_t w)(x,t) = \int_0^t w(x,t')\,dt'$ for functions w depending on $x \in \Omega$ and (or) $t \in (0,T)$.

We use the classical Lebesgue space $L^q(G)$ and its anisotropic version $L^{q,r}(Q)$ with the norm $\|w\|_{L^{q,r}(Q)} = \big\| \|w\|_{L^q(\Omega)} \big\|_{L^r(0,T)}$ for $q,r \in [1,\infty]$. Let $\frac{1}{q'} + \frac{1}{q} = 1$.

We introduce the class (not the space) $\mathcal{N}(Q)$ of functions $w \in L^\infty(Q)$ such that $w > 0$, $1/w \in L^\infty(Q)$, and $D_t w \in L^2(Q)$ and set $\big| w \big|_{\mathcal{N}(Q)} = \|w\|_{L^\infty(Q)} + \|1/w\|_{L^\infty(Q)} + \|D_t w\|_{L^2(Q)}$. We also use the Banach spaces $V_q(Q)$ and $W(Q)$ with the norms $\|w\|_{V_q(Q)} = \|w\|_{L^{q,\infty}(Q)} + \|Dw\|_{L^q(Q)}$ and $\|w\|_{W(Q)} = \|w\|_{L^2(Q)} + \|D_t w\|_{L^2(Q)} + \|Dw\|_{L^{2,\infty}(Q)}$. Let $S_2^{1,1}W(Q)$ be the Hilbert space (of functions w having the dominating mixed derivative $DD_t w \in L_2(Q)$) with the norm $\|w\|_{S_2^{1,1}W(Q)} = \big(\|w\|^2_{L^2(Q)} + \|Dw\|^2_{L^2(Q)} + \|D_t w\|^2_{L^2(Q)} + \|DD_t w\|^2_{L^2(Q)}\big)^{1/2}$. Note that $S_2^{1,1}W(Q) \subset W(Q) \subset C(\overline{Q})$, moreover, the latter embedding is compact.

Let $V[0,T]$ be the space of functions of the bounded variation on $[0,T]$.

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2 Weak solutions and their global existence

Let us consider a system of quasilinear differential equations of a perfect polytropic gas 1D–flow

$$D_t\eta = Du, \tag{2.1}$$

$$D_tu = D\sigma + g[x_e], \quad \sigma = \nu\rho Du - p, \quad p = k\rho\theta, \quad \rho = 1/\eta, \tag{2.2}$$

$$c_V D_t\theta = D\pi + \sigma Du + f[x_e], \quad \pi = \lambda\rho D\theta, \tag{2.3}$$

$$D_t x_e = u \tag{2.4}$$

in the domain Q. Here the sought functions η, u, θ, and x_e depend on the Lagrangian mass coordinates $(x,t) \in Q$, and $F[x_e](x,t) = F(x_e(x,t),x,t)$ for $F = g, f$. We supplement the system of equations with the initial and boundary conditions

$$(\eta, u, \theta, x_e)|_{t=0} = (\eta^0, u^0, \theta^0, x_e^0) \quad \text{on} \quad \Omega; \tag{2.5}$$

$$u|_{x=0} = u_0(t) \text{ (or } \sigma|_{x=0} = -p_0(t)), \; u|_{x=X} = u_X(t) \text{ (or } \sigma|_{x=X} = -p_X(t)), \tag{2.6}$$

$$-\pi|_{x=0} = \chi_0(t), \quad \pi|_{x=X} = \chi_X(t) \tag{2.7}$$

with $x_e^0(x) = \int_0^x \eta^0(\xi)\,dx$ and $0 < t < T$. We denote the initial-boundary value problem (IBVP) (2.1)–(2.7) by $\mathcal{P}$. Let us set $m = 1,2,3,$ or 4, if $u|_{x=0,X}; u|_{x=0}$ and $\sigma|_{x=X}; \sigma|_{x=0}$ and $u|_{x=X}$; or $\sigma|_{x=0,X}$, respectively, are prescribed on the boundary. We suppose that those boundary functions u_0, u_X, p_0, and p_X that do not used in the mth boundary condition (2.6) are equal to zero.

We recall the physical meaning of the quantities involved: η, u, and θ are the specific volume, velocity, and absolute temperature of a gas; x_e is the Eulerian coordinate; ρ and p are the density and pressure; σ and $-\pi$ are the stress and heat flux; g and f are the densities of the mass forces and heat sources; u_α, p_α, and χ_α, for $\alpha = 0, X$, are the velocity of piston, external pressure, and external heat flux; X is the total mass of the gas.

Now we list our conditions on the data. Let $N > 1$ be an arbitrarily large parameter and $u_\Gamma = (u_0, u_X)$, $p_\Gamma = (p_0, p_X)$, and $\chi_\Gamma = (\chi_0, \chi_X)$ be the boundary data pairs.

C_1. The coefficients $\nu, k, c_V, \lambda \in L^\infty(\Omega)$ are such that $N^{-1} \le \nu \le N$, $0 \le k \le N$, $N^{-1} \le c_V \le N$, and $N^{-1} \le \lambda \le N$.

C_2. The initial functions $\eta^0 \in L^\infty(\Omega)$, $u^0 \in L^2(\Omega)$, and $\theta^0 \in L^1(\Omega)$ are such that $N^{-1} \le \eta^0 \le N$, $\theta^0 > 0$, and $\|e^0\|_{L^1(\Omega)} + \|s^0\|_{L_1(\Omega)} \le N$, where $e^0 = \frac{1}{2}(u^0)^2 + c_V\theta^0$ and $s^0 = k\log\eta^0 + c_V\log\theta^0$ are the initial energy and entropy.

C_3. The free terms g and f are Carathéodory functions on $\mathbf{R} \times Q$; moreover, $|g(\chi,\cdot)| \le \overline{g}(\cdot)$ and $0 \le f(\chi,\cdot) \le \overline{f}(\cdot)$ on Q for all $\chi \in \mathbf{R}$, where $\|\overline{g}\|_{L^{2,1}(Q)} + \|\overline{f}\|_{L^1(Q)} \le N$.

C_4. The boundary data $u_\Gamma \in V[0,T]$, $p_\Gamma \in L^\infty(0,T)$, and $\chi_\Gamma \in L^1(0,T)$ are such that $p_0 \ge 0$, $p_X \ge 0$, $\chi_0 \ge 0$, $\chi_X \ge 0$, and $\|u_\Gamma\|_{V[0,T]} + \|p_\Gamma\|_{L^\infty(0,T)} + \|\chi_\Gamma\|_{L^1(0,T)} \le N$. Furthermore, for all $\tau \in (0,T)$ and almost all $t \in (0, T-\tau)$, the inequality holds

$$p_\alpha(t+\tau) - p_\alpha(t) \le a_\tau(t) \int_t^{t+\tau} p_\alpha(t')\,dt', \quad \alpha = 0, X \tag{2.8}$$

with some $a_\tau \in L^1(0,T)$, $a_\tau \geq 0$, and $\sup_{0<\tau<T} \|a_\tau\|_{L^1(0,T)} \leq N$. Finally, if both $u|_{x=0} = u_0$ and $u|_{x=X} = u_X$ are prescribed on the boundary, then $N^{-1} \leq V(t) = \|\eta^0\|_{L^1(\Omega)} + I_t(u_X - u_0)$ on $(0,T)$.

Notice that the possibility to take variable discontinuous rather than constant coefficients allows to consider some contact problems [20] as well. The unusual one-sided condition (2.8) is broad enough. For example, if $0 < N_1^{-1} \leq p_\alpha$ and $\mathrm{var}_{[0,T]}\, p_\alpha \leq N_2$, then it is satisfied (with $N = N_1 N_2$). On the other hand, it is evidently satisfied for nonincreasing $p_\alpha \geq 0$ (with $a_\tau \equiv 0$).

In a standard manner a vector-function $z = (\eta, u, \theta, x_e) \in \mathcal{N}(Q) \times V_2(Q) \times V_1(Q) \times S_2^{1,1}W(Q)$ will be called a *weak solution* to problem $\mathcal{P}$, if $\theta > 0$ and the following properties are valid:

(a) equations (2.1) and (2.4) are satisfied in $L^2(Q)$;

(b) the integral identities

$$\textstyle\int_Q (-uD_t\varphi + \sigma D\varphi)\, dxdt = \int_\Omega u^0 \varphi|_{t=0}\, dx + \int_Q g[x_e]\varphi\, dxdt \qquad (2.9)$$

$$+ \textstyle\int_0^T (p_0\varphi|_{x=0} - p_X\varphi|_{x=X})\, dt,$$

$$\textstyle\int_Q (-c_V\theta D_t\psi + \pi D\psi)\, dxdt = \qquad (2.10)$$

$$= \textstyle\int_\Omega c_V\theta^0\psi|_{t=0}\, dx + \int_Q (\sigma Du + f[x_e])\,\psi\, dxdt$$

$$+ \textstyle\int_0^T (\chi_0\psi|_{x=0} + \chi_X\psi|_{x=X})\, dt$$

hold, with σ and π defined in (1.2) and (1.3), for all $\varphi, \psi \in C^1(\overline{Q})$ such that $\varphi\big|_{t=T} = \psi\big|_{t=T} = 0$; moreover, $\varphi|_{x=\alpha} = 0$ provided that $u|_{x=\alpha}$ is prescribed on the boundary $x = \alpha$, for $\alpha = 0, X$;

(c) the initial conditions $\eta|_{t=0} = \eta^0$ and $x_e|_{t=0} = x_e^0$ and the boundary conditions $u|_{x=0} = u_0$ and $u|_{x=X} = u_X$ (if prescribed) are satisfied in the sense of traces.

Now we are in position to state global with respect to time and data existence theorem for a weak solution to problem $\mathcal{P}$. Let $K(N)$ (possibly with indexes) be positive functions of N; they can also depend on X, T.

Theorem 2.1 *1. Suppose that conditions C_1–C_4 are valid. Then problem $\mathcal{P}$ has a weak solution $z = (\eta, u, \theta, x_e)$ satisfying the estimates*

$$|\,\eta\,|_{\mathcal{N}(Q)} + \|u\|_{V_2(Q)} + \|\theta\|_{V_1(Q)} + \|x_e\|_{S_2^{1,1}W(Q)} \leq K(N),$$

$$\|\theta\|_{L^{q_0,r_0}(Q)} + \|D\theta\|_{L^{q_1,r_1}(Q)} \leq K_\varepsilon(N),$$

$$\|\log\theta\|_{L^{1,\infty}(Q)} + \|D\log\theta\|_{L^2(Q)} \leq K(N),$$

$$\|u\|_{2,2}^{\langle 0,1/2\rangle} + \|\theta\|_{2,1}^{\langle 0,1/2\rangle} \leq K(N)$$

for all $q_0, r_0 \in [1,\infty]$ and $q_1, r_1 \in [1,2]$ such that $(2q_0)^{-1} + r_0^{-1} = (1+\varepsilon)/2$ and $(2q_1)^{-1} + r_1^{-1} = (1+\varepsilon)/4$ with any $\varepsilon \in (0,1)$.

2. This solution is such that $\eta \in C([0,T]; L^\infty(\Omega))$, $u \in C([0,T]; L^2(\Omega))$, and $\theta \in C([0,T]; L^1(\Omega))$, while satisfying the initial conditions (2.5) in the sence of the spaces in question. Moreover, the following estimates for the primitive in t functions of σ and π hold:

$$\|I_t\sigma\|_{W(Q)} \leq K(N), \quad \|I_t\pi\|_{C([0,T];\, W^{1,1}(\Omega))} \leq K(N). \qquad (2.11)$$

3. If, in addition, $N^{-1} \le \theta^0$, then $[K(N)]^{-1} \le \theta$, i.e. the temperature is strictly positive.

Here along with the spaces introduced above, the seminorm $\|w\|_{2,r}^{\langle 0,1/2\rangle} = \sup_{0<\tau<T} \tau^{-1/2}\|w(x,t+\tau) - w(x,t)\|_{L^{2,r}(Q_{T-\tau})}$ (in the anisotropic Nikol'skii space $H_{2,r}^{0,1/2}(Q)$), the space $C([0,T];B)$ of continuous on $[0,T]$ functions with values in a Banach space B, and the Sobolev space $W^{1,1}(\Omega)$ have been used.

It is essential both from the mathematical and physical point of view that under conditions C_1–C_4 in our definition of a weak solution to problem $\mathcal{P}$ integral identity (2.11) can be equivalently replaced, see [8] and [4], by the following identity involving the energy $e = \frac{1}{2}u^2 + c_V\theta$:

$$\int_\Omega [-eD_t\psi + (\sigma u + \pi)D\psi]\,dxdt = \int_\Omega e^0\psi|_{t=0}\,dx + \int_Q (g[x_e]u + f[x_e])\psi\,dxdt \quad (2.12)$$
$$+(S)\int_0^T (I_t\sigma)|_{x=0}d(u_0\psi|_{x=0}) - (S)\int_0^T (I_t\sigma)|_{x=X}\,d(u_X\psi|_{x=X})$$
$$+\int_0^T \left[(p_0u|_{x=0} + \chi_0)\psi|_{x=0} - (p_Xu|_{x=X} - \chi_X)\psi|_{x=X}\right]dt$$
$$\forall\psi \in C^1(\overline{Q}), \quad \psi|_{t=T} = 0,$$

where the integrals marked by (S) are the Riemann-Stieltjes ones. This identity is a weak form of the well-known energy equation but we call attention to the nonstandard terms containing the boundary data u_0 and u_X. They are correctly defined as $u_0, u_X \in V[0,T]$ and $I_t\sigma \in C(\overline{Q})$, see the next section. The advantage of identity (2.12) is the absence of quadratic nonlinearities in Du; so it rather than (2.11) has to be used for weak passages to the limit.

To prove Theorem 2.1, at least two approaches can be distinguished. First, one can smooth all the data, get a global regular solution [11], prove a requived collection of estimates for the solution independently of the small smoothing parameter, say $\delta > 0$, and then pass to the limit as $\delta \to 0$. Such an approach was represented in [3]. Its disadvantage is that the full proof of Theorem 2.1 becomes rather lengthy as the proof in [11] is not simple (and is given only for constant coefficients and $g = f = 0$). Aside from the long sequence of nontrivial *a priori* estimates, it contains the auxiliary local existence theorem proved by some Galerkin method. In our opinion, this method is purely auxiliary and does not have any value for practical computations.

Second, one can construct a suitable approximate semidiscrete method (finite difference in x), prove required estimates for its solution independently of the discretization parameter $h > 0$, and pass to the limit as $h \to 0$ globally in time. Such an approach was represented in [6] and is described in the rest of the section.

Its merit is that the proof becomes self-contained, moreover, after deducing additional regularity results (such as Theorem 4.2 below) one can get another proof of a global regular solution [11] existence. Once the approach was successful, the semidiscrete method takes on a new meaning for practical computations. Notice

that usually a proof of *a priori* estimates for semidiscrete method is more complicated and lengthy than that for the original problem. We managed to make them more close to each other as only derivatives and integrals (but not finite differences and sums) are used.

Of course, the other approaches of proving Theorem 2.1 are also possible. For example, one can use a suitable finite difference scheme instead of the semidiscrete method. The last approach was represented in [10] even for a system of equations more complicated than (2.1)–(2.4). But unfailingly the construction of such a scheme is nontrivial and the proof of *a priori* estimates becomes much more cumbersome. So the result is assigned more to numerical analysis than differential equations.

Let us describe a semidiscrete method for solving problem $\mathcal{P}$. We need additional notation. Set $h = X/n$ for $n \geq 2$ and introduce two grids, with nodes $x_i = ih$, $0 \leq i \leq n$, and $x_{i-1/2} = \left(i - \frac{1}{2}\right) h$, $1 \leq i \leq n$, respectively. Let $\Omega_{1/2} = [0, x_1)$, $\Omega_{i-1/2} = (x_{i-1}, x_i)$ for $2 \leq i \leq n-1$, and $\Omega_{n-1/2} = (x_{n-1}, X]$. Let also $\Omega_0 = [0, x_{1/2})$, $\Omega_i = (x_{i-1/2}, x_{i+1/2})$ for $1 \leq i \leq n-1$, and $\Omega_n = (x_{n-1/2}, X]$.

We introduce the spaces of piecewise constant functions $S^h = \{w \mid w(x) = w_i$ on $\Omega_i,\ 0 \leq i \leq n\}$ and $S^h_{1/2} = \{v \mid v(x) = v_{i-1/2}$ on $\Omega_{i-1/2},\ 1 \leq i \leq n\}$. For $w \in S^h$, let $\widehat{w} \in C(\overline{\Omega})$, $\widehat{w}$ be linear on $\Omega_{i-1/2}$, $1 \leq i \leq n$, and $\widehat{w}(x_i) = w_i$, $0 \leq i \leq n$. For $v \in S^h_{1/2}$, let $\widehat{v} \in C(\overline{\Omega})$, $\widehat{v}$ be linear on Ω_i, $0 \leq i \leq n$, and $\widehat{v}(x_{i-1/2}) = v_{i-1/2}$, $1 \leq i \leq n$; moreover, $\widehat{v}$ be defined by some way at the points $x = 0, X$ and, if the other way is not prescribed, $\widehat{v}(0) = v_{1/2}$ and $\widehat{v}(X) = v_{n-1/2}$. Let also $(I_h w)(x) = \int_0^{x_{i-1/2}} w(\xi)\, d\xi$ on $\Omega_{i-1/2}$, $1 \leq i \leq n$, for $w \in S^h$ and $(I_h v)(x) = \int_0^{x_i} v(\xi)\, d\xi$ on Ω_i, $0 \leq i \leq n$, for $v \in S^h_{1/2}$. We set $\Omega^{h;1} = (x_{1/2}, x_{n-1/2})$, $\Omega^{h;2} = (x_{1/2}, X)$, $\Omega^{h;3} = (0, x_{n-1/2})$, and $\Omega^{h;4} = \Omega$.

For $y \in L^1(\Omega)$, let $s^h y \in S^h$ (resp. $s^h_{1/2} y \in S^h_{1/2}$) be equal on Ω_i, $0 \leq i \leq n$, (resp. on $\Omega_{i-1/2}$, $1 \leq i \leq n$) to the mean value of y on this set.

We approximate the system of equations (2.1)–(2.4) as follows:

$$D_t \eta^h = D\widehat{u}^h \text{ in } Q; \tag{2.13}$$

$$D_t u^h = D\widehat{\sigma}^h + g^h \text{ in } Q^{h;m};\ \sigma^h = \nu \rho^h D\widehat{u}^h - p^h,\ p^h = k\rho^h \theta^h,\ \rho^h = 1/\eta^h \text{ in } Q; \tag{2.14}$$

$$c_V D_t \theta^h = D\widehat{\pi}^h + \sigma^h D\widehat{u}^h + f^h \text{ in } Q; \quad \pi^h = \lambda \widetilde{\rho}^h D\widehat{\theta}^h \text{ in } Q^{h;1}; \tag{2.15}$$

$$D_t x_e^h = u^h \quad \text{in } Q, \tag{2.16}$$

where $g^h = s^h g[x_e^h]$, $f^h = s^h_{1/2} f[x_e^h]$, $\widetilde{\rho}^h = 1/s^h \eta^h$, and $Q^{h;m} = \Omega^{h;m} \times (0, T)$; for simplicity, we take the constant coefficients. The sought functions are $\eta^h, \theta^h \in W^{1,1}(0, T; S^h_{1/2})$ and $u^h, x_e^h \in W^{1,1}(0, T; S^h)$; for example, $\eta^h \in W^{1,1}(0, T; S^h_{1/2})$ means that $\eta^h, D_t \eta^h \in L^1(0, T; S^h_{1/2})$. It is supposed that $\eta^h > 0$, $\theta^h > 0$.

We also approximate the initial and boundary conditions (2.5)–(2.7) as follows:

$$(\eta^h, u^h, \theta^h, x_e^h)|_{t=0} = (\eta^{h,0}, u^{h,0}, \theta^{h,0}, x_e^{h,0}), \tag{2.17}$$

$$u^h|_{x=0} = u_0^{(\tau)} \ (\text{or } \widehat{\sigma}^h|_{x=0} = -p_0^{(\tau)}), \quad u^h|_{x=X} = u_X^{(\tau)} \ (\text{or } \widehat{\sigma}^h|_{x=X} = -p_X^{(\tau)}), \tag{2.18}$$

$$-\pi^h|_{x=0} = \chi_0, \quad \pi^h|_{x=X} = \chi_X. \tag{2.19}$$

Here $\eta^{h,0} = s^h_{1/2}\eta^0$, $\theta^{h,0} = s^h_{1/2}\theta^0$, $x_e^{h,0} = I_h\eta^{h,0}$; moreover, $u^{h,0} = s^h u^0$ but if $u^h|_{x=0} = u_0^{(\tau)}$ (resp. $u^h|_{x=X} = u_X^{(\tau)}$) is prescribed on the boundary, then we redefine $u^{h,0}(x) = u_0^{(\tau)}(0)$ on Ω_0 (resp. $u^{h,0}(x) = u_X^{(\tau)}(0)$ on Ω_n). Further, $\varphi^{(\tau)}(t) = \tau^{-1}\int_t^{t+\tau}\varphi(t')\,dt'$ $(0 < \tau \le N^{-2})$ for $\varphi = u_\alpha, p_\alpha$, and $u_\alpha(t) = u_\alpha(T)$ and $p_\alpha(t) = 0$ for $t > T$, $\alpha = 0, X$. We denote problem (2.13)–(2.19) by $\mathcal{P}^h$.

The following theorem is a semidiscrete counterpart of Theorem 2.1 (namely, claims 1 and 3 in it).

Theorem 2.2 *1. Suppose that conditions C_1–C_4 are valid. Then problem $\mathcal{P}^h$ has a solution satisfying the estimates*

$$| \eta^h |_{N(Q)} + \|\widehat{u}^h\|_{V_2(Q)} + \|\widehat{\theta}^h\|_{V_1(Q)} + \|\widehat{x}_e\|_{S_2^{1,1}W(Q)} \le K(N), \tag{2.20}$$

$$\|\theta^h\|_{L^{q_0,r_0}(Q)} + \|D\widehat{\theta}^h\|_{L^{q_1,r_1}(Q)} \le K_\varepsilon(N), \tag{2.21}$$

$$\|\log\theta^h\|_{L^{1,\infty}(Q)} + \|D\widehat{\log\theta^h}\|_{L^2(Q)} \le K(N), \tag{2.22}$$

$$\|\widehat{u}^h\|_{2,2}^{\langle 0,1/2\rangle} + \|\theta^h\|_{2,1}^{\langle 0,1/2\rangle} \le K(N) \tag{2.23}$$

for q_0, r_0 and q_1, r_1 specified in Theorem 2.1.

2. If, in addition, $N^{-1} \le \theta^0$, then $[K(N)]^{-1} \le \theta^h$.

In fact problem $\mathcal{P}^h$ is the Cauchy problem for a system of ordinary differential equiations (after the algebraic elimination of the prescribed boundary data). The existence of its solution on a sufficiently small segment of time $0 \le t \le T_0$ $(T_0 > 0)$ follows from the classical Carathéodory theorem. The possibility to continue the solution to the whole of the segment $[0, T]$ preserving the properties $\eta^h > 0$ and $\theta^h > 0$ is justified by *a priori* estimate (2.20) and the estimate $\theta^h \ge \delta^h = \exp(-K(N)/h) > 0$, see (2.22). So the main point in the proof of Theorem 2.2 is the derivation of *a priori* estimates in claims 1 and 2.

Let us briefly describe the stages of the proof of Theorem 2.2. Note that equation (2.13) can be rewritten (see the definition of σ^h) in the following form:

$$D_t\eta^h = (1/\nu)\sigma^h\eta^h + (k/\nu)\theta^h. \tag{2.24}$$

Let $V^h(t) = \|\eta^h(\cdot, t)\|_{L^1(\Omega)}$ be the semidiscrete total volume of gas.

Lemma 2.1 *If both $u^h|_{x=0}$ and $u^h|_{x=X}$ are prescribed on the boundary, then $V^h = \|\eta^0\|_{L^1(\Omega)} + I_t(u_X^{(\tau)} - u_0^{(\tau)})$ and the two-sided estimate $(2N)^{-1} \le V^h \le K(N)$ holds.*

Introduce the function $v_m^h = u^h - u^h_{\Gamma,m}$, where $u^h_{\Gamma,1} = (1-\ell^h)u_0^{(\tau)} + \ell^h u_X^{(\tau)}$ with $\ell^h = I_h\eta^h/V^h$, $u^h_{\Gamma,2} = u_0^{(\tau)}$, $u^h_{\Gamma,3} = u_X^{(\tau)}$, $u^h_{\Gamma,4} = 0$. The following formulas hold:

$$D_t u^h_{\Gamma,1} = (1-\ell^h)D_t u_0^{(\tau)} + \ell^h D_t u_X^{(\tau)} + d_V v_1^h, \quad D\widehat{u}^h_{\Gamma,1} = d_V\eta^h \tag{2.25}$$

with $d_V = (u_X^{(\tau)} - u_0^{(\tau)})/V^h$.

Lemma 2.2 *The folllowing conservation laws hold:*

$$D_t \int_\Omega \frac{1}{2}(v_m^h)^2\,dx + \int_\Omega \sigma^h D\widehat{u}^h\,dx = \Gamma^h + \int_\Omega (g^h - D_t u_{\Gamma,m}^h)v_m^h\,dx, \quad (2.26)$$

$$D_t \int_\Omega \left[\frac{1}{2}(v_m^h)^2 + c_V\theta^h\right] dx = \Gamma^h + \int_\Omega \left[(g^h - D_t u_{\Gamma,m}^h)v_m^h + f^h\right] dx + \chi_0 + \chi_X, \quad (2.27)$$

where $\Gamma^h = \int_\Omega \sigma^h D\widehat{u}_{\Gamma,m}^h\,dx + (\widehat{\sigma}^h v_m^h)|_{x=X} - (\widehat{\sigma}^h v_m^h)|_{x=0}$; *moreover,*

$$\Gamma^h = D_t \textstyle\int_\Omega d_V \nu\eta^h\,dx - \int_\Omega \left[(D_t d_V)\nu\eta^h + d_V k\theta^h\right] dx \quad \textit{for } m = 1,$$

$$\Gamma^h = D_t \textstyle\int_\Omega \sigma_{\Gamma,m}^{(\tau)}\eta^h\,dx - \int_\Omega (D_t\sigma_{\Gamma,m}^{(\tau)})\eta^h\,dx \quad \textit{for } m = 2,3,$$

$$\Gamma^h = D_t \textstyle\int_\Omega \sigma_{\Gamma,4}^{(\tau)}\eta^h\,dx - \int_\Omega (D_t\sigma_{\Gamma,4}^{(\tau)})\eta^h\,dx - X^{-1}\int_\Omega (p_X^{(\tau)} - p_0^{(\tau)})u^h\,dx \textit{ for } m = 4$$

with $\sigma_{\Gamma,2} = -p_X$, $\sigma_{\Gamma,3} = -p_0$, *and* $\sigma_{\Gamma,4} = -(1 - \frac{x}{X})p_0 - \frac{x}{X}p_X$.

The proof is more cumbersome than in the well-known (differential) case of homogeneous boundary conditions. Formulas (2.24) and (2.25) are involved in it.

Proposition 2.1 *The energy estimate* $\|u^h\|_{L^{2,\infty}(Q)} + \|\theta^h\|_{L^{1,\infty}(Q)} \le K(N)$ *holds.*

The estimate is derived from law (2.27) and Lemma 2.1.

By solving equation (2.24) with respect to η^h, we derive the important formula

$$\eta^h = \left[\eta^{0,h} + (k/\nu)I_t(\theta^h/\gamma^h)\right]\gamma^h \quad \text{with} \quad \gamma^h = \exp\big((1/\nu)I_t\sigma^h\big). \quad (2.28)$$

Lemma 2.3 *The following uniform estimate holds:* $\|I_t\sigma^h\|_{L^\infty(Q)} \le K(N)$.

If $\widehat{\sigma}^h$ is prescribed at least on one of the boundaries, for example, for $x = 0$, then $I_t\sigma^h = I_h(u^h - u^{0,h} - I_t g^h) - I_t p_0^{(\tau)}$ and the proof of the estimate is rather simple. Otherwise, the proof becomes more complicated.

Corollary 2.1 *The following two-sided estimates for* η^h *and* V^h *holds*

$$[K(N)]^{-1} \le \eta^h \le K(N)(1 + I_t\theta^h), \quad [K(N)]^{-1} \le V^h \le K(N).$$

The first estimate follows from formula (2.28). The second estimate is a consequence of the first one (see also Lemma 2.1).

Lemma 2.4 *The law of nondecreasing entropy holds in the following form:*

$$D_t \textstyle\int_\Omega (c_V \log\theta^h + k\log\eta^h)\,dx = \int_\Omega [-\lambda\widetilde{\rho}^h(D\widehat{\theta}^h)D(\widehat{1/\theta^h}) +$$
$$+(\nu\rho^h(D\widehat{u}^h)^2 + f^h)/\theta^h]\,dx + \chi_0/\theta^h|_{x=0} + \chi_X/\theta^h|_{x=X} \ge 0.$$

This lemma is a semidiscrete counterpart of the well-known law.

Corollary 2.2 *The following auxiliary estimates hold:*

$$\|\beta^h\|_{L^1(0,T)} \le K(N), \quad \|\log\theta^h\|_{L^{1,\infty}(Q)} + \|(\widetilde{\rho}^h)^{1/2}D\widehat{\log\theta^h}\|_{L^2(Q)} \le K(N)$$

with $\beta^h = \int_\Omega [-\widetilde{\rho}^h(D\widehat{\theta}^h)D(\widehat{1/\theta^h})]\,dx$.

Let us formulate the last auxiliary result.

Lemma 2.5 *The following estimates for θ^h hold:*

$$\|\theta^h\|_{L^\infty(\Omega)} \le X^{-1}\|\theta^h\|_{L^1(\Omega)} + \|D\widehat{\theta}^h\|_{L^1(\Omega)} \le K(N)\big[\beta^h\|\eta^h\|_{L^\infty(\Omega)} + 1\big].$$

Now it is possible to derive the estimates in Theorem 2.2.

Proposition 2.2 *The following estimates for η^h, θ^h, and p^h hold:*

$$[K(N)]^{-1} \le \eta^h \le K(N), \quad \|\widehat{\theta}^h\|_{V_1(Q)} \le K(N), \quad \|p^h\|_{L^2(Q)} \le K(N).$$

The first estimate follows from Corollaries 2.1 and 2.2 and Lemma 2.5. Then the last two estimates hold in virtue of Lemma 2.5 and the energy estimate.

Proposition 2.3 *The estimate $\|D\widehat{u}^h\|_{L^2(Q)} \le K(N)$ together with estimates (2.21), (2.23) for θ^h and u^h hold. Moreover, claim 2 of Theorem 2.2 is valid.*

The estimate $\|D\widehat{u}^h\|_{L^2(Q)} \le K(N)$ is proved with the help of law (2.26). The rest of the estimates is really an application of bounds for a semidiscrete method to linear parabolic IBVP with discontinuous coefficients, the right-hand side from $L^{2,1}(Q)$ or $L^1(Q)$, and the initial function from $L^2(\Omega)$ or $L^1(\Omega)$. See, for instance, Proposition 3.1 below as a differential version, and [6] for details.

Finally, the estimates $\|D_t\eta^h\|_{L^2(Q)} \le K(N)$, $\|\widehat{x}_e^h\|_{S_2^{1,1}W(Q)} \le K(N)$, and (2.22) easily follows from equations (2.13),(2.16), and Corollary 2.2.

Theorem 2.1 (claims 1 and 3) is derived from Theorem 2.2 by the limit transition as $h \to 0$. Notice that estimates (2.20), (2.23), and a counterpart of the first estimate (2.11) are used when establishing the convergence (for a subsequence) of u^h in $L^2(Q)$, θ^h in $L^1(Q)$, and $I_t\sigma^h$ in $C([0,T];L^\infty(\Omega))$. Then by formula (2.28) the convergence of η^h in $C([0,T];L^q(\Omega))$ for all $1 \le q < \infty$ is established. The limit transition in semidiscrete counterparts of integral identities (2.9) and (2.12) is also used. See [6] for details.

3 The uniqueness and Lipschitz continuous dependence on data for weak solutions

It is well-known that, for quasilinear evolution problems in continuum mechanics, the problem of uniqueness for weak solutions can be more complicated than that of existence. In a sense this is true for problem $\mathcal{P}$ under consideration. Originally the uniqueness theorem was proved, for not so weak solutions as in Section 2, under the additional assumptions on the initial data $u^0 \in L^\infty(\Omega)$ and $\theta^0 \in L^4(\Omega)$, see [2]. Only in [21] the uniqueness theorem was established, by another approach, under natural conditions on the data.

The statement of the latter uniqueness theorem (but not its proof) is simple. We introduce the additional condition:

C_5. There exist (weak) derivatives $D_\chi g$ and $D_\chi f$ such that

$$\big\|\|D_\chi g\|_{L^\infty(-a,a)}\big\|_{L^{2,1}(Q)} + \big\|\|D_\chi f\|_{L^\infty(-a,a)}\big\|_{L^1(Q)} \le C(a) \quad \text{for all} \quad a > 1.$$

Theorem 3.1 *Suppose that conditions C_1–C_5 are valid. Then the weak solution to problem $\mathcal{P}$ from the class $\mathcal{N}(Q) \times V_2(Q) \times V_1(Q) \times S_2^{1,1}W(Q)$ (defined in Section 2) is unique.*

Of course, condition C_5 is a version of the standard local Lipschitz condition on g and f with respect to 'nonlinear' argument χ; if g and f do not depend on χ, then it is automatically satisfied.

In fact Theorem 3.1 is proved together with a theorem on the Lipschitz continuous dependence on data (namely, the initial data, boundary data, and free terms) for the weak solutions. To state the latter theorem, we need additional notation.

Let us consider two problems, $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$, of the form $\mathcal{P}$. In the problem $\mathcal{P}^{(\ell)}$, $\ell = 1, 2$, the initial data η^0, u^0, and θ^0; the boundary data u_Γ, p_Γ, and χ_Γ; and the free terms g and f we supply with the superscript (ℓ) and write down as $\eta^{0,(\ell)}$, $u_\Gamma^{(\ell)}$, $g^{(\ell)}$, etc. The weak solution to the problem $\mathcal{P}^{(\ell)}$ we denote by $z^{(\ell)} = (\eta^{(\ell)}, u^{(\ell)}, \theta^{(\ell)}, x_e^{(\ell)})$.

Introduce the operation of taking a difference Δ such that $\Delta\varphi = \varphi^{(1)} - \varphi^{(2)}$; then $\Delta\eta^0 = \eta^{0,(1)} - \eta^{0,(2)}$, $\Delta u_\Gamma = u_\Gamma^{(1)} - u_\Gamma^{(2)}$, $\Delta\eta = \eta^{(1)} - \eta^{(2)}$, etc. Set $\Delta F[x_e^{(2)}] = F^{(1)}[x_e^{(2)}] - F^{(2)}[x_e^{(2)}]$ for $F = g, f$. Let us use the expansion $\Delta f[x^{(2)}] = f' + f''$ with some $f', f'' \in L^1(Q)$ (for example, with $f' = 0$ or $f'' = 0$).

To define negative norms for the free terms, we denote by $\mathcal{F}^m(Q)$ the space of functions $F \in L^1(Q)$ with the finite norm

$$\|F\|_{\mathcal{F}^m(Q)} = \sup \frac{\int_Q F\varphi\, dxdt}{\|\varphi\|_{V_2(Q)}},$$

where the supremum is taken over all $\varphi \in V_2(Q)$, $\varphi \not\equiv 0$ such that $\varphi|_{x=\alpha} = 0$ provided that $u|_{x=\alpha}$ is prescribed on the boundary $x = \alpha$, for $\alpha = 0, X$; it is supposed that $F\varphi \in L^1(Q)$ for all φ specified. Introduce also the primitive functions and the mean value $(Iv)(x) = \int_0^x v(\xi)\, d\xi$, $(I^*v)(x) = \int_x^X v(\xi)\, d\xi$, and $\langle v\rangle_\Omega = X^{-1}\int_0^X v(x)\, dx$ and the operators $I^{\langle 1\rangle}v = Iv - \langle Iv\rangle_\Omega$, $I^{\langle 2\rangle}v = I^*v$, $I^{\langle 3\rangle}v = Iv$, $I^{\langle 4\rangle}v = I(v - \langle v\rangle_\Omega)$ for $v \in L^1(\Omega)$. Note that the following inequalities hold

$$\|F\|_{\mathcal{F}^m(Q)} \le \|I^{\langle m\rangle}F\|_{L^2(Q)} \quad \text{for} \quad m = 1, 2, 3,$$
$$\|F\|_{\mathcal{F}^m(Q)} \le \|I^{\langle m\rangle}F\|_{L^2(Q)} + \sqrt{X}\|\langle F\rangle_\Omega\|_{L^1(0,T)} \quad \text{for} \quad m = 4$$

provided that $F \in L^1(Q)$ and $I^{\langle m\rangle}F \in L^2(Q)$. On the other hand, we have

$\|F\|_{\mathcal{F}^m(Q)} \le c\|F\|_{L^{q,r}(Q)}$ for all $q \in [1, 2], r \in [1, 4/3]$ such that $(2q)^{-1} + r^{-1} = 5/4$.

Theorem 3.2 *Suppose that conditions C_1–C_5 are valid for the data of both problems $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$. Then the following estimate for the norm of difference between the weak solutions to these problems by the sum of norms of the corresponding differences between their data holds:*

$$| \Delta\eta |_{\mathcal{N}(Q)} + \|\Delta u\|_{V_2(Q)} + \|\Delta\theta\|_{L^{q,r}(Q)} + \|\Delta x_e\|_{S_2^{1,1}W(Q)}$$
$$\le K_{\varepsilon_1,\,\varepsilon}(N)\big(\|\Delta\eta^0\|_{L^\infty(\Omega)} + \|\Delta u^0\|_{L^2(\Omega)} + \|\Delta\theta^0\|_{L^1(\Omega)}$$

$$+\|\Delta u_\Gamma\|_{V[0,T]} + \|\Delta p_\Gamma\|_{L^{4/3}(0,T)} + \|\Delta\chi_\Gamma\|_{L^1(0,T)}$$
$$+\|\Delta g[x_e^{(2)}]\|_{\mathcal{F}^m(Q)} + \|I^{\langle m\rangle} I_t \Delta g[x_e^{(2)}]\|_{L^\infty(Q)}$$
$$+\|I^{\langle 4\rangle} f'\|_{L^{3/(2+\varepsilon_1-\varepsilon)}(Q)} + \|\langle f'\rangle_\Omega\|_{L^1(0,T)} + \|f''\|_{L^1(Q)}\big)$$

for all $q \in [2,\infty)$, $r \in [2,4)$ *such that* $(2q)^{-1} + r^{-1} = (1+\varepsilon_1)/2$ *and* $q = \infty$, $r = 2/(1+\varepsilon_1)$ *with any* $0 < \varepsilon < \varepsilon_1 < 1/2$.

In the statement of this theorem, the weak solutions to problems $\mathcal{P}^{(1)}$ and $\mathcal{P}^{(2)}$ are unique according to Theorem 3.1.

Let us describe briefly how to derive Theorems 3.1 and 3.2. The derivation relies heavily on a few propositions concerning estimates for various weak solutions to linear parabolic IBVP with discontinuous data. The statement of these Propositions 3.1–3.3 will be postponed until the end of the section.

First, for *any* weak solution to problem $\mathcal{P}$, we deduce the following formulas

$$\eta = \big[\eta^0 + (k/\nu) I_t(\theta/\gamma)\big]\gamma \quad \text{with} \quad \gamma = \exp\big((1/\nu) I_t\sigma\big), \tag{3.2}$$
$$I_t\sigma = I^{\langle m\rangle}(u - u^0 - I_t g[x_e]) + I_t\sigma_{\Gamma,m}, \tag{3.3}$$

where $\sigma_{\Gamma,1} = \langle\sigma\rangle_\Omega$ whereas $\sigma_{\Gamma,m}$ for $m = 2,3,4$ were introduced in Lemma 2.2 above. Such formulas [11] are very important as they do not contain derivatives of the sought functions. Moreover, under the assumption

$$|\!|\!| \eta |\!|\!|_{\mathcal{N}(Q)} + \|u\|_{V_2(Q)} + \|\theta\|_{V_1(Q)} + \|x_e\|_{C(\overline{Q})} \le \overline{N} \tag{3.4}$$

with some $\overline{N} > N$, we prove the estimates

$$\|I_t\sigma\|_{C(\overline{Q})} \le K_1(\overline{N}), \tag{3.5}$$
$$\|\theta\|_{L^{q_0,r_0}(Q)} + \|D\theta\|_{L^{q_1,r_1}(Q)} \le K_2(\overline{N}) \tag{3.6}$$

for q_0, r_0 and q_1, r_1 specified in Theorem 2.1. Estimate (3.5) follows from formula (3.3) whereas estimate (3.6) follows from Proposition 3.1 (see below) on $V_1(Q)$–solutions to parabolic IBVP applied to equation (2.3).

Second, we take any solutions $z^{(1)}$ to problem $\mathcal{P}^{(1)}$ and $z^{(2)}$ to problem $\mathcal{P}^{(2)}$. Without loss of generality, one can suppose that they satisfy assumption (3.4). We establish a collection of auxiliary estimates for the components of the difference $\Delta z = (\Delta\eta, \Delta u, \Delta\theta, \Delta x_e)$ between the solutions with any $0 < t \le T$. Namely, by using formulas (3.2), (3.3), and estimate (3.5), we get the following estimate for $\Delta\eta$:

$$\|\Delta\eta\|_{L^\infty(Q_t)} \le K_3(\overline{N})\big(\|\Delta u\|_{V_1(Q_t)} + \|\Delta\theta\|_{L^{\infty,1}(Q_t)} + \|b\Delta x_e\|_{L^1(Q_t)} \tag{3.7}$$
$$+\Delta_{1,m} + \|I^{\langle m\rangle}\Delta u^0\|_{L^\infty(\Omega)} + \|I_t\Delta p_\Gamma\|_{L^\infty(0,T)}\big),$$

where
$b = b(x,t)$, $\|b\|_{L^{2,1}(Q)} \le K_4(\overline{N})$, $\Delta_{1,m} = \|\Delta\eta^0\|_{L^\infty(\Omega)} + \|I^{\langle m\rangle} I_t \Delta g[x_e^{(2)}]\|_{L^\infty(Q)}$.

By applying the energy estimate for $V_2(Q)$–solution to parabolic IBVP [17], [5] to such a problem for Δu, we obtain the following estimate for Δu:

$$\|\Delta u\|_{V_2(Q_t)} \le K_5(\overline{N})\big(\|d\Delta\eta\|_{L^{\infty,2}(Q_t)} + \|\Delta\theta\|_{L^2(Q_t)} + \|b\Delta x_e\|_{L^{2,1}(Q_t)} + \Delta_{2,m}\big), \tag{3.8}$$

where $d = d(t)$, $\|d\|_{L^2(0,T)} \le K_6(\overline{N})$, and $\Delta_{2,m} = \|\Delta u^0\|_{L^2(\Omega)} + \|\Delta u_\Gamma\|_{V[0,T]}$ $+\|\Delta p_\Gamma\|_{L^{4/3}(0,T)} + \|\Delta g[x_e^{(2)}]\|_{\mathcal{F}^m(Q)}$.

Then, by using estimates (3.7) and (3.8), taking into account the simple estimate

$$\|\Delta x_e\|_{C(\overline{\Omega})} \le \|\Delta\eta^0\|_{L^1(\Omega)} + \|\Delta\eta\|_{L^1(\Omega)} + X^{-1}\|\Delta u\|_{L_1(Q_t)} \tag{3.9}$$

as well as the Gronwall lemma, we have the combined estimate for $\Delta\eta$ and Δu:

$$\|\Delta\eta\|_{L^\infty(Q_t)} + \|\Delta u\|_{V_2(Q_t)} \le K_7(\overline{N})\big(\|\Delta\theta\|_{L^{\infty,1}(Q_t)} + \|\Delta\theta\|_{L^2(Q_t)} + \Delta_{1,m} + \Delta_{2,m}\big). \tag{3.10}$$

Next, by utilizing Propositions 3.1 and 3.3 (see below), the latter one about $L^2(Q)-$solution to parabolic IBVP, to such a problem for $\Delta\theta$ and taking into account estimates (3.6) and (3.9), we deduce the following estimate for $\Delta\theta$:

$$\begin{aligned}\|\Delta\theta\|_{L^{q,r}(Q_t)} \le K_{8,\varepsilon_1,\varepsilon}(\overline{N})\big(&\|\Delta\eta\|_{L^\infty(Q_t)} + \|\Delta u\|_{V_2(Q_t)} \\ &+\|\Delta\theta\|_{L^{\infty,1}(Q_t)} + \|\Delta\theta\|_{L^2(Q_t)} + \Delta_{3,m}\big),\end{aligned} \tag{3.11}$$

with q, r and $\varepsilon_1, \varepsilon$ specified in Theorem 3.2 and $\Delta_{3,m} = \|\Delta\eta^0\|_{L^\infty(\Omega)} + \|\Delta\theta^0\|_{L^1(\Omega)} +$ $+\|\Delta\chi_\Gamma\|_{L^1(0,T)} + \|I^{(4)}f'\|_{L^{3/(2+\varepsilon_1-\varepsilon)}(Q)} + \|\langle f'\rangle_\Omega\|_{L^1(0,T)} + \|f''\|_{L^1(Q)}$.

The last estimate (3.11) is crucial in the proof. It is very important that one can take $q = \infty$ and some $r > 1$ as well as $q = 2$ and some $r > 2$ there.

Finally, by combining estimates (3.10) and (3.11), and applying Lemma 3.1 (see below) that plays the role of the Gronwall lemma and also using estimate (3.9), we obtain the following estimate for Δz:

$$\begin{aligned}\|\Delta\eta\|_{L^\infty(Q)} + \|\Delta u\|_{V_2(Q)} + \|\Delta\theta\|_{L^{q,r}(Q)} + \|\Delta x_e\|_{C(\overline{Q})}& \\ \le K_{9,\varepsilon_1,\varepsilon}(\overline{N})(\Delta_{1,m} + \Delta_{2,m} + \Delta_{3,m}).&\end{aligned} \tag{3.12}$$

By taking into account equations (2.1) and (2.4), one can easily replace $L^\infty(Q)$ by $\mathcal{N}(Q)$ and $C(\overline{Q})$ by $S_2^{1,1}W(Q)$. Applying this estimate in the particular case that $z^{(1)}$ and $z^{(2)}$ are simply two solutions to problem $\mathcal{P}$, we derive Theorem 3.1. Then, applying (3.12) in the general case and utilizing Theorems 2.1 and 3.1, we can take $\overline{N} = K(N)$ and finish the proof of Theorem 3.2.

The rest of the section is devoted to important auxiliary statements mentioned above. We consider the following linear parabolic IBVP

$$D_t(\alpha v) = D(\beta Dv - \psi) + F \quad \text{in} \quad Q,$$
$$v|_{t=0} = v^0(x) \quad \text{on} \quad \Omega,$$
$$v|_{x=0} = 0 \ (\text{or } (\beta Dv - \psi)|_{x=0} = s_0(t)),\ v|_{x=X} = 0 \ (\text{or } (\beta Dv - \psi)|_{x=X} = s_X(t)).$$

We denote this IBVP by $\mathcal{L}$. Let $\alpha, \beta \in L^\infty(Q)$, $N^{-1} \le \alpha \le N$, $N^{-1} \le \beta \le N$ and $\|D_t\beta\|_{L^{q_\beta,r_\beta}(Q)} \le N$ for some $q_\beta, r_\beta \in [1,\infty]$ such that $(2q_\beta)^{-1} + r_\beta^{-1} = 1$; a smoothness (or continuity) of α and β in x does not supposed.

Proposition 3.1 *Suppose that $F \in L^1(Q)$, $v^0 \in L^1(\Omega)$, $s_0, s_X \in L^1(0,T)$, $\psi = 0$, and $\|D_t\alpha\|_{L^{q_\alpha,r_\alpha}(Q)} \le N$ for some q_α, r_α such that $(2q_\alpha)^{-1} + (r_\alpha)^{-1} = 1 - \varepsilon_\alpha/2$*

with some $\varepsilon_\alpha \in (0,1)$. Then problem $\mathcal{L}$ has a unique weak solution v from $V_1(Q)$. Moreover, $v, \alpha v \in C([0,T]; L^1(\Omega))$ and the following estimate holds:

$$\|v\|_{C([0,T];L^1(\Omega))} + \|v\|_{L^{q_0,r_0}(Q)} + \|Dv\|_{L^{q_1,r_1}(Q)} \tag{3.13}$$
$$\le K_\varepsilon(N)\big(\|v^0\|_{L^1(\Omega)} + \|F\|_{L^1(Q)} + \|s_0\|_{L^1(0,T)} + \|s_X\|_{L^1(0,T)}\big)$$

for q_0, r_0 and q_1, r_1 specified in Theorem 2.1 but with $\varepsilon \in (0, \varepsilon_\alpha]$. In addition, the initial condition $v|_{t=0} = v^0$ is satisfied in the sense of the space $C([0,T]; L^1(\Omega))$.

Note that the definition of $V_1(Q)$–solution is quite similar to the standard definition of $V_2(Q)$–solution, see [17] (and also integral identity (2.11)).

Let us treat the problem $\mathcal{L}^0$, which is a special case of problem $\mathcal{L}$ for $\psi \in L^{6/5}(Q)$, $F = 0$, $v^0 = 0$, and $s_0 = s_X = 0$. First, in accordance with [17], we call a function $v \in L^2(Q)$ a *weak solution from $L^2(Q)$*, if it satisfies the integral identity

$$-\int_Q v[\alpha D_t\varphi + D(\beta D\varphi)]\,dxdt = \int_Q \psi D\varphi\,dxdt$$

for all $\varphi \in H^1(Q)$ such that $\beta D\varphi \in V_2(Q)$, $\varphi|_{t=T} = 0$, and also $\varphi|_{x=\alpha} = 0$ (resp. $(\beta D\varphi)|_{x=\alpha} = 0$) provided that $v|_{x=\alpha} = 0$ (resp. $(\beta Dv)|_{x=\alpha} = 0$) is prescribed on the boundary $x = \alpha$, for $\alpha = 0, X$.

Problem $\mathcal{L}^0$ has a unique solution v from $L^2(Q)$, and the following estimate holds

$$\|v\|_{L^2(Q)} \le K(N)\|\psi\|_{L^{6/5}(Q)}. \tag{3.14}$$

But for our purposes it is essential to have an additional estimates for v in the case $\psi \in L^p(Q)$, $p \in (6/5, 2)$. Let us state such a results; for simplicity, let $\alpha = \alpha(x)$.

First, we bound v in $V_p(Q)$ for $p > 2$ and Dv in $L^p(Q)$ for $p < 2$ but, in general, only for p close to 2 in the both cases.

Proposition 3.2 *1. Let v be $V_1(Q)$–solution to problem $\mathcal{L}^0$. There exists $\overline{p} = 2 + \delta_0$ with $\delta_0 = 1/K_1(N)$ such that, for all $p \in [2, \overline{p}]$, if $\psi \in L^p(Q)$, then $v \in V_p(Q) \cap C([0,T]; L^p(\Omega))$ and the following estimate holds:*

$$\|v\|_{V_p(Q)} \le K(N)\|\psi\|_{L^p(Q)}.$$

2. Let v be $L^2(Q)$–solution to problem $\mathcal{L}^0$ and $\underline{p} = \max\{\overline{p}', 6/5\}$. For any $p \in [\underline{p}, 2]$, if $\psi \in L^p(Q)$, then $Dv \in L^p(Q)$ and the following estimate holds:

$$\|v\|_{L^2(Q)} + \|Dv\|_{L^p(Q)} \le K(N)\|\psi\|_{L^p(Q)}. \tag{3.15}$$

Note that claim 1 is really valid *without* the above condition on $D_t\beta$.

Next, we bound v in $L^{q,r}(Q)$ and separately in $L^{\infty,r}(Q)$.

Proposition 3.3 *Let v be $L^2(Q)$–solution to problem $\mathcal{L}^0$.*

1. Let $q, r \in [2, \infty]$, $(2q)^{-1} + r^{-1} = (1-\delta)/2$, $|\delta| < 1/2$, and $0 < \varepsilon < 1/2 - \delta$. If $\psi \in L^p(Q)$ with $p = 3/(2 - \delta - \varepsilon)$, then $L^{q,r}(Q)$–estimate for v holds:

$$\|v\|_{L^{q,r}(Q)} \le K_{1,\delta,\varepsilon}(N)\|\psi\|_{L^p(Q)}.$$

2. Let $r = 1 + (p_0')^{-1}$ *with* $p_0 \in (\underline{p}, 2)$ *and* $0 < \varepsilon_0 < 1 - r^{-1}$. *If* $\psi \in L^p(Q)$ *with* $p = \left[(5/6)(1 - r^{-1} - \varepsilon_0) + p_0^{-1}(r^{-1} + \varepsilon_0)\right]^{-1}$, *then* $L^{\infty,r}(Q)$*-estimate for* v *holds:*

$$\|v\|_{L^{\infty,r}(Q)} \le K_{2,r,\varepsilon_0}(N)\|\psi\|_{L^p(Q)}.$$

The following result can replace the Gronwall lemma in some situations.

Lemma 3.1 *Let* $1 \le s_i < r_i \le \infty$ *for* $i = 1, \ldots, n$. *Suppose that a collection of functions* $Y_i \in L^{r_i}(0,T)$, $i = 1, \ldots, n$, *satisfies the inequality*

$$\sum_{1\le i\le n} \|Y_i\|_{L^{r_i}(0,t)} \le C_0 + \sum_{1\le i\le n} \|b_i Y_i\|_{L^{s_i}(0,t)} \quad \text{for all} \quad t \in (0,T],$$

where $C_0 = const \ge 0$ *and* $\|b_i\|_{L^{\mu\overline{r}_i}(0,T)} \le N$ *with* $\overline{r}_i = (s_i^{-1} - r_i^{-1})^{-1}$, $i = 1, \ldots, n$, *and some* $\mu > 1$. *Then the following estimate holds:*

$$\sum_{1\le i\le n} \|Y_i\|_{L^{r_i}(0,T)} \le K^0(N) C_0,$$

where $K^0(N)$ *depends only on* N, T, *and* μ.

All the Propositions 3.1–3.3 and Lemma 3.1 (as well as a result much more general than estimate (3.14)) were proved in [21]. Here we notice only the following. The proof of estimate (3.13) exploits $L^\infty(Q)$–estimate of the well-known type [17] (see also [4]) for the solution to the problem dual to problem $\mathcal{L}$. Claim 1 of Proposition 3.2 is deduced by some nonstandard duality argument [5] from the related Kruzhkov theorem [16] concerning $W^{2,1;p}(Q)$–solvability of nondivergent parabolic IBVP with the coefficients in $L^\infty(Q)$. Claim 2 follows from claim 1 by the standard duality argument. The estimates in Proposition 3.3 are derived from estimate (3.14), the energy estimate $\|v\|_{V_2(Q)} \le K(N)\|\psi\|_{L^2(Q)}$, and estimate (3.15) by methods of the theory of real interpolation of Banach spaces [12]. Lemma 3.1 is a simple technical result.

4 Regularity of weak solutions

Let us turn to regularity theorems for a weak solution to problem $\mathcal{P}$ (under the same conditions C_1 on the coefficients).

First, we analyze the internal regularity for u and θ together with σ and π (for simplicity, with respect to t only) under the same broad conditions C_2 on the initial data. Let $\zeta \in H^1(0,T)$ be a cut-off function such that $\|D_t\zeta\|_{L^2(0,T)} \le N$ and $\zeta|_{t=0} = 0$ and $V_2^{\langle 1,1/2\rangle}(Q) = V_2(Q) \cap H_{2,2}^{0,1/2}(Q)$ be the Banach space with the norm $\|w\|_{V_2^{\langle 1,1/2\rangle}(Q)} = \|w\|_{V_2(Q)} + \|w\|_{2,2}^{\langle 0,1/2\rangle}$.

Theorem 4.1 *1. Suppose that conditions* C_1–C_4 *as well as the conditions*

$$\|\zeta\overline{g}\|_{L^2(Q)} + \|\zeta^2\overline{f}\|_{L^2(Q)} \le N, \tag{4.2}$$

$$\|D_t(\zeta u_\Gamma)\|_{L^{4/3}(0,T)} + \|\zeta p_\Gamma\|_{V[0,T]} + \|\zeta^2\chi_\Gamma\|_{V[0,T]} \le N \tag{4.3}$$

are valid. Then, for the weak solution in Theorem 2.1, the additional regularity $D_t(\zeta u), D_t(\zeta^2\theta) \in L^2(Q)$ *and* $\zeta\sigma, \zeta^2\pi \in V_2^{\langle 1,1/2\rangle}(Q)$ *take place and the estimates hold:*

$$\|D_t(\zeta u)\|_{L^2(Q)} + \|D_t(\zeta^2\theta)\|_{L^2(Q)} \le K(N),$$
$$\|\zeta\sigma\|_{V_2^{\langle 1,1/2\rangle}(Q)} + \|\zeta^2\pi\|_{V_2^{\langle 1,1/2\rangle}(Q)} \le K(N).$$

As a consequence, $\zeta u, \zeta^2\theta \in C(\overline{Q})$ *and* $D(\zeta u), D(\zeta^2\theta) \in L^{q,r}(Q)$ *and the estimates*

$$\|\zeta u\|_{C(\overline{Q})} + \|\zeta^2\theta\|_{C(\overline{Q})} \le K(N), \quad \|D(\zeta u)\|_{L^{q,r}(Q)} + \|D(\zeta^2\theta)\|_{L^{q,r}(Q)} \le K(N)$$

hold for all $q, r \in [1,\infty]$ *such that* $(2q)^{-1} + r^{-1} = 1/4$. *Moreover, the equations*

$$D_t(\zeta u) = D(\zeta\sigma) + \zeta g[x_e] + (D_t\zeta)u,$$
$$c_V D_t(\zeta^2\theta) = D(\zeta^2\pi) + \zeta^2\sigma Du + \zeta^2 f[x_e] + c_V(D_t(\zeta^2))\theta$$

are satisfied in $L^2(Q)$.

2. If, in addition, the following conditions on $\overline{g}$, $\overline{f}$, *and* u_Γ *are valid:*

$$\|\zeta^2\overline{g}\|_{L^{2q_2,2r_2}(Q)} + \|\zeta^3\overline{f}\|_{L^{2q_3,2r_3}(Q)} \le N, \quad \|D_t(\zeta^2 u_\Gamma) - (D_t(\zeta^2))u_\Gamma\|_{L^{r_\varepsilon}(0,T)} \le N$$

for some $q_i, r_i \in [1,\infty], i = 2,3,$ *such that* $(2q_i)^{-1} + r_i^{-1} = 1-\varepsilon$ *and for* $r_\varepsilon = 2/(1-\varepsilon)$, *with some* $\varepsilon \in (0,1]$, *then* $\zeta^2\sigma, \zeta^3\pi \in L^\infty(Q)$ *and the estimate holds*

$$\|\zeta^2\sigma\|_{L^\infty(Q)} + \|\zeta^3\pi\|_{L^\infty(Q)} \le K_\varepsilon(N).$$

To illuminate Theorem 4.1, one can fix $0 < t_1 < t_2 \le T$ and arbitrarily small $\delta \in (0, t_1)$, choose $\zeta(t) = 0$ on $[0, t_1 - \delta]$ and $\zeta(t) = 1$ on $J = (t_1, t_2)$, and apply the theorem in the case $T = t_2$. Set $Q' = \Omega \times J$ and $Q'_\delta = \Omega \times J_\delta$ with $J_\delta = (t_1 - \delta, t_2)$. Then, under conditions C_1–C_4 and $\overline{g}, \overline{f} \in L^2(Q'_\delta)$, $D_t u_\Gamma \in L^{4/3}(J_\delta)$, $p_\Gamma, \chi_\Gamma \in V(\overline{J}_\delta)$, the regularity properties $D_t u, D_t\theta \in L^2(Q')$, $\sigma, \pi \in V_2^{\langle 1,1/2\rangle}(Q')$, $u, \theta \in C(\overline{Q}')$, and $Du, D\theta \in L^{q,r}(Q')$ are valid, for q, r specified in Theorem 4.1. Moreover, the first equations (2.2) and (2.3) are satisfied not only in the weak sense but in $L^2(Q')$ as well. If, in addition, for q_i, r_i, and r_ε as in Theorem 4.1, $\overline{g} \in L^{2q_2,2r_2}(Q'_\delta)$, $\overline{f} \in L^{2q_3,2r_3}(Q'_\delta)$, and $D_t u_\Gamma \in L^{r_\varepsilon}(J_\delta)$, then $\sigma, \pi \in L^\infty(Q')$. We emphasize that the properties $D\sigma = D(\nu\rho Du - p) \in L^2(Q')$ and $D\pi = D(\lambda\rho D\theta) \in L^2(Q')$ are somewhat striking because due to the lack of regularity of η^0 and ν, k, λ, generally speaking, the functions ρ, Du, p, and $D\theta$ *do not have* the weak derivative D in Q'.

Now it is natural to consider the regularity up to the moment $t = 0$. In this connection we introduce a new definition of solution. A vector-function $z = (\eta, u, \theta, x_e) \in \mathcal{N}(Q) \times H^1(Q) \times H^1(Q) \times S_2^{1,1}W(Q)$ will be called an *almost regular weak solution* to problem $\mathcal{P}$, if $D\sigma, D\pi \in L^2(Q)$, $\theta > 0$, and all the equations (2.1)–(2.4) are satisfied in $L^2(Q)$ and the initial and boundary conditions (2.5)–(2.7) are satisfied in the sense of traces.

In this definition, the properties of η are the same as above. In contrast with the definition of a regular weak solution in [11], we do not suppose that $D\eta \in L^{2,\infty}(Q)$

and $D^2u, D^2\theta \in L^2(Q)$ (or even that these derivatives exist). But this is no barrier for us to consider $D\sigma, D\pi \in L^2(Q)$ and the first equations (2.2), (2.3) in $L^2(Q)$.

Theorem 4.2 *1. Suppose that conditions C_1–C_4 as well as conditions*

$$\|Du^0\|_{L^2(\Omega)} + \|D\theta^0\|_{L^2(\Omega)} \leq N, \quad \|\overline{g}\|_{L^2(Q)} + \|\overline{f}\|_{L^2(Q)} \leq N,$$
$$\|D_t u_\Gamma\|_{L^{4/3}(0,T)} + \|p_\Gamma\|_{V[0,T]} + \|\chi_\Gamma\|_{V[0,T]} \leq N$$

are valid. Let also the conjugacy condition $u_0(0^+) = u^0(0)$ (resp. $u_X(0^+) = u^0(X)$) be valid provided that $u|_{x=0} = u_0$ (resp. $u|_{x=X} = u_X$) is prescribed on the boundary. Then the weak solution in Theorem 2.1 is an almost regular one satisfying the estimates

$$\|D_t u\|_{L^2(Q)} + \|D_t\theta\|_{L^2(Q)} \leq K(N), \quad \|\sigma\|_{V_2^{(1,1/2)}(Q)} + \|\pi\|_{V_2^{(1,1/2)}(Q)} \leq K(N),$$
$$\|u\|_{C(\overline{Q})} + \|\theta\|_{C(\overline{Q})} \leq K(N), \quad \|Du\|_{L^{q,r}(Q)} + \|D\theta\|_{L^{q,r}(Q)} \leq K(N)$$

for all $q, r \in [1, \infty]$ such that $(2q)^{-1} + r^{-1} = 1/4$.

2. If, in addition, the more strong conditions on u^0, θ^0, $\overline{g}, \overline{f}$, and u_Γ are valid:

$$\|Du^0\|_{L^\infty(\Omega)} + \|D\theta^0\|_{L^\infty(\Omega)} \leq N,$$
$$\|\overline{g}\|_{L^{2q_2,2r_2}(Q)} + \|\overline{f}\|_{L^{2q_3,2r_3}(Q)} \leq N, \; \|D_t u_\Gamma\|_{L^{r_\varepsilon}(0,T)} \leq N$$

for some $q_i, r_i \in [1, \infty], i = 2, 3$, such that $(2q_i)^{-1} + r_i^{-1} = 1 - \varepsilon$ and for $r_\varepsilon = 2/(1-\varepsilon)$, with some $\varepsilon \in (0, 1]$, then the estimate in $L^\infty(Q)$–norm holds:

$$\|\sigma\|_{L^\infty(Q)} + \|\pi\|_{L^\infty(Q)} \leq K(N).$$

3. Let the conditions in Item 2 together with the continuity properties $\sigma^0 \equiv \nu\rho^0 Du^0 - k\rho^0\theta^0 \in C(\overline{\Omega})$, $\pi^0 \equiv \lambda\rho^0 D\theta^0 \in C(\overline{\Omega})$ (where $\rho^0 = 1/\eta^0$), and $p_\Gamma \in C[0, T]$ be valid. Let also the conjugacy condition $\sigma^0(0) = -p_0(0)$ (resp. $\sigma^0(X) = -p_X(0)$) be valid provided that $\sigma|_{x=0} = -p_0$ (resp. $\sigma|_{x=X} = -p_X$) is prescribed on the boundary. Then $\sigma, \pi \in C(\overline{Q})$, i.e. σ and π are continuous on $\overline{Q}$.

We emphasize once again that, in the above theorems, for the coefficients ν, k, c_V, λ and initial function η^0, it is sufficient to belong to $L^\infty(\Omega)$ and to be strictly positive.

To prove Theorems 4.1 and 4.2, we establish their counterparts for semidiscrete problem $\mathcal{P}^h$ and then pass to the limit as $h \to 0$; see [9] for details.

Remarks

1. The results of Section 2 were proved [6] in the case of the more general than (2.7) boundary conditions $(-\pi + \mu_0\theta)|_{x=0} = \chi_0(t)$ and $(\pi + \mu_X\theta)|_{x=X} = \chi_X(t)$, where $\mu_0 \geq 0, \mu_X \geq 0$, and $\|\mu_0\|_{L^s(0,T)} + \|\mu_X\|_{L^s(0,T)} \leq N$ for some $s > 2$. Moreover, the first and (or) second of the last boundary conditions can be replaced by the conditions $\theta|_{x=0} = \theta_0(t)$ and (or) $\theta|_{x=X} = \theta_X(t)$, respectively, where $\theta_0 > 0, \theta_X > 0$, and $\|\theta_0\|_{V[0,T]} + \|\theta_X\|_{V[0,T]} \leq N$, see [1].

2. Other theorems concerning the Lipschitz continuous dependence on data, for the constant coefficients and not so weak solutions as in Section 2, can be found in [2], [22].

3. Weak solutions to the combustion problem for a viscous heat-conducting gas 1D-equations were considered in [24].

4. The homogenization problem for rapidly oscillating periodic nonsmooth coefficients and initial data as well as weak solutions to the limiting integro-differential Bakhvalov-Eglit system (which is more general than system (2.1)–(2.4)) were studied in [7],[8].

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Regularity Criteria of the Axisymmetric Navier-Stokes Equations

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Abstract

In this note we present results on the regularity criteria for the axisymmetric weak solutions of the three dimensional Navier-Stokes equations with nonzero swirl. In particular we find that the integrability of single component of vorticity or velocity fields, in terms of norms with zero scaling dimension give sufficient conditions for the regularity of weak solutions. To obtain these criteria we derive new *a priori* estimates for the axisymmetric smooth solutions of the Navier-Stokes equations.

1 Introduction

In this paper, we are concerned with the initial value problem of the Navier-Stokes equations in $\mathbb{R}^3 \times [0,T)$.

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u = -\nabla p, \tag{1}$$

$$\text{div } u = 0, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

where $u = (u_1, u_2, u_3)$ with $u = u(x,t)$, and $p = p(x,t)$ denote unknown fluid velocity and scalar pressure, respectively, while u_0 is a given initial velocity satisfying div $u_0 = 0$. In the above we denote

$$(u \cdot \nabla)u = \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j}, \qquad \Delta u = \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_j^2}.$$

For simplicity we assume that there is no external force term in the right hand side of (1). Inclusion of the external force does not provide essential difficulty in our

works of this paper. We recall that *the Leray-Hopf weak solution* of the Navier-Stokes equations is defined as a vector field $u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ satisfying div $u = 0$ in the sense of distribution, the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2 \quad \text{a.e. } t \in [0,T]$$

and

$$\int_0^T \int_{\mathbb{R}^3} (u \cdot \phi_t + (u\cdot\nabla)\phi\cdot u + u \cdot \Delta\phi) dx dt + \int_{\mathbb{R}^3} u_0(x)\cdot \phi(x,0)dx = 0,$$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times [0,T))^3$ with div $\phi = 0$.
For given $u_0 \in L^2(\mathbb{R}^3)$ with div $u_0 = 0$ in the sense of distribution, a global weak solution u to the Navier-Stokes equations was constructed by Leray[14] and Hopf[11], which we call the Leray-Hopf weak solution. For review on the theory of the Leray-Hopf weak solution see [8] and [24]. Regularity of such Leray-Hopf weak solutions is one of the most outstanding open problems in the mathematical fluid mechanics. For further discussion below we introduce the Banach space $L_T^{\alpha,\gamma}$, equipped with the norm

$$\|u\|_{L_T^{\alpha,\gamma}} = \left(\int_0^T \|u(t)\|_\gamma^\alpha dt\right)^{\frac{1}{\alpha}},$$

where

$$\|u(t)\|_\gamma = \begin{cases} \left(\int_{\mathbb{R}^3} |u(x,t)|^\gamma dx\right)^{\frac{1}{\gamma}} & \text{if } 1 \le \gamma < \infty \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3} |u(x,t)| & \text{if } \gamma = \infty \end{cases}.$$

In particular, we use the same norm for scalar function $u = u(x)$ and the vector function $u(x) = (u_1(x), u_2(x), u_3(x))$. Taking the *curl* operation of (1), we obtain the evolution equation of the vorticity $\omega = \text{curl } u$,

$$\frac{\partial \omega}{\partial t} - \Delta\omega + (u\cdot\nabla)\omega - (\omega\cdot\nabla)u = 0. \tag{4}$$

As an approach to the regularity problem Serrin[22](See also[20]) studied the regularity criterion of the Leray-Hopf weak solutions, and obtained that if a weak solution u belongs to $L_T^{\alpha,\gamma}$ where $\frac{2}{\alpha} + \frac{3}{\gamma} < 1$, $2 < \alpha \le \infty$, $3 < \gamma \le \infty$, then u is regular, and becomes a smooth in space variables on $(0,T]$. After Serrin's work, there are many improvements and developments regarding the study of regularity criterion.(See e.g. [9], [10], [12], and [23].) It is found, in particular, the Leray-Hopf weak solution becomes smooth in (x,t) if $u \in L_T^{\alpha,\gamma}$ with $\frac{2}{\alpha} + \frac{3}{\gamma} \le 1$, $2 \le \alpha \le \infty$, $3 < \gamma \le \infty$. On the other hand, Beirão da Veiga[1] obtained the regularity criterion by imposing the integrability of the gradient of the velocity. Recently, one of the authors[3] improved Beirão da Veiga's result by imposing the same integrability condition on two components of the vorticity. Very Recently, Neustupa et al[18] obtained regularity criterion by imposing integrability of single component of velocity field. The integrability condition here, however, is stronger than the Serrin's one, and is not optimal

in the sense of scaling considerations. Moreover, the weak solution concerned in [18] is not the Leray-Hopf weak solutions, but the so called suitable weak solutions introduced first in [2].

We are concerned here with the regularity criteria of axisymmetric weak solutions of the Navier-Stokes equations. By an axisymmetric solution of the Navier-Stokes equations we mean a solution of the equations of the form

$$u(x,t) = u^r(r,x_3,t)e_r + u^\theta(r,x_3,t)e_\theta + u_3(r,x_3,t)e_3$$

in the cylindrical coordinate system, where we used the basis

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0),\ e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0),\ e_3 = (0,0,1),\ r = \sqrt{x_1^2 + x_2^2}.$$

In the above u^θ is called the *swirl* component of the velocity u. For the axisymmetric solutions, we can rewrite the equation (1) and (2) as follows.

$$\begin{aligned}
&\frac{\tilde{D}u^r}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u^r + \frac{1}{r^2}u^r - \frac{1}{r}u^\theta u^\theta + \partial_r p = 0,\\
&\frac{\tilde{D}u^\theta}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u^\theta + \frac{1}{r^2}u^\theta + \frac{1}{r}u^\theta u^r = 0,\\
&\frac{\tilde{D}u_3}{Dt} - (\partial_r^2 + \partial_3^2 + \frac{1}{r}\partial_r)u_3 + \partial_3 p = 0,\\
&\partial_r(ru^r) + \partial_3(ru_3) = 0,
\end{aligned}$$

where we denote

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + u^r\partial_r + u_3\partial_3.$$

In the following, we will also use the notation for the axisymmetric vector field u

$$\tilde{u} = u^r e_r + u_3 e_3,$$

and

$$\tilde{\nabla} = (\partial_r, \partial_3).$$

For the axisymmetric vector field u, we can compute the vorticity $\omega = \text{curl}\, u$ as follows.

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega_3 e_3,$$

where

$$\omega^r = -\partial_3 u^\theta, \quad \omega_3 = \partial_r u^\theta + \frac{u^\theta}{r}, \quad \omega^\theta = -\partial_r u_3 + \partial_3 u^r.$$

For the study of axisymmetric solutions of the Navier-Stokes equations *without swirl*, Ukhovski and Yudovich [25] proved the existence of generalized solutions, uniqueness and the regularity. Recently, Leonardi, Málek, Nečas and Porkoný [13] gave a refined proof of the results in [25]. In the related case of helical symmetry, Mahalov, Titi and Leibovich [15] proved the global existence of the strong solution. For the

axisymmetric Navier-Stokes equations with nonzero swirl component, however, the regularity problem is still open. For the axisymmetric Euler equations with swirl, the azimuthal component of the vorticity in the cylindrical coordinates alone controls the blow-up of the velocity(See [5] and for the other studies on the axisymmetric solutions of the Euler equations, see [4], [16], [17], [21], [25] and references therein.).

2 Main Results

Our main results are the following four theorems. The detailed proofs of which are in [6], and will be published elsewhere.

Theorem 1 *Let u be an axisymmetric weak solution of the Navier-Stokes equations with $u_0 \in H^2(\Omega)$ div $u_0 = 0$ and $Q_T = \Omega \times [0,T)$, where Ω is bounded domain or $\mathbb{R}^3$. If ω^θ is in $L_T^{\alpha,\gamma}(Q_T)$ where α and γ satisfies $\frac{3}{2} < \gamma < \infty$, $1 < \alpha \le \infty$ and $\frac{2}{\alpha} + \frac{3}{\gamma} \le 2$, then the weak solution u is smooth in $Q'_T = \Omega' \times (0,T)$ where $\Omega' \subset\subset \Omega$.*

Remark 1. Comparing with the result in [3], we find that Theorem 1 is an obvious improvement of the corresponding criterion theorem in [3], which is, in turn, improvement of [1] for the axisymmetric case.
The following theorem provides us the available *a priori* estimate for ω^θ, which is new for the axisymmetric solutions to the knowledge of the authors.

Theorem 2 *If u is an axisymmetric smooth solution of the Navier-Stokes equations with initial data $u_0 \in L^2(\mathbb{R}^3)$ with div $u_0 = 0$ satisfying $r^3\omega_0^\theta \in L^2(\mathbb{R}^3)$ and $ru_0^\theta \in L^4(\mathbb{R}^3)$, then $r^3\omega^\theta \in L_T^{\infty,2} \cap L^2(0,T;H^1(\mathbb{R}^3))$.*

For a given axisymmetric function f on $\mathbb{R}^3$, we define the $\tilde{L}^\gamma$ norm by

$$\|f\|_{\tilde{L}^\gamma} = \left(\int_{-\infty}^{\infty}\int_0^{\infty} |f(r,x_3)|^\gamma dr dx_3\right)^{\frac{1}{\gamma}}.$$

Theorem 3 *Let u be an axisymmetric weak solution of the Navier-Stokes equations with the initial data u_0 satisfying $ru_0^\theta \in L^4(\mathbb{R}^3)$, $r^3\omega_0^\theta \in L^2(\mathbb{R}^3)$ and $u_0 \in H^2(\mathbb{R}^3)$.*

(i) *If u^r and u^θ is in $L^\alpha(0,T;L^\gamma(drdx_3))$ where $2 < \gamma < \infty$, $2 < \alpha \le \infty$ and $\frac{1}{\alpha} + \frac{1}{\gamma} \le \frac{1}{2}$, then the solution is smooth in $\mathbb{R}^3 \times (0,T)$.*

(ii) *If ω^θ is in $L^\alpha(0,T;L^\gamma(drdx_3))$ where $1 < \gamma < \infty$, $1 < \alpha \le \infty$ and $\frac{1}{\alpha} + \frac{1}{\gamma} \le 1$, then the solution is smooth in $\mathbb{R}^3 \times (0,T)$.*

Remark 2. One of the interesting consequence of Theorem 3 (ii) is that if $\omega^\theta \in L^2(0,T;L^2(drdx_3))$, then the solution becomes smooth. By the energy inequality for weak solutions, however, we know that $\omega \in L^2(0,T;L^2(rdrdx_3))$ for the Leray-Hopf weak solutions. Thus there is a discrepancy between the available estimate and the criterion in the region only near the axis of symmetry.

We note that the norm $\|u\|_{L^\alpha(0,T;L^\gamma(drdx_3))}$ has zero scaling dimension if $\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{1}{2}$, and $\|\omega\|_{L^\alpha(0,T;L^\gamma(drdx_3))}$ has zero scaling dimension if $\frac{1}{\alpha}+\frac{1}{\gamma}=1$. The above criteria are optimal in this sense.
As an immediate corollary of Theorem 3 (ii), we can reprove the following result on the regularity of axisymmetric weak solutions in the case without swirl, which was obtained previously by Ukhovski and Yudovich [25] and Leonardi et al. [13] using different argument.

Corollary 1 *Suppose u_0 is an axisymmetric initial data without swirl, satisfying the hypothesis of Theorem 3. Then there exists a smooth solution on $\mathbb{R}^3 \times (0,T)$.*

Next, we obtain regularity criteria in terms of a single component of velocity field.

Theorem 4 *Let $\delta > 0$ be given, and set $\Gamma_\delta = \{x \in \mathbb{R}^3 \mid r < \delta\}$. Suppose that u is an axisymmetric weak solution, and satisfies the hypothesis of Theorem 3 and one of the following conditions.*

Either

(i) $\frac{u^r}{r} \in L^\alpha(0,T;L^\gamma(\Gamma_\delta))$ *where α and γ satisfy $\frac{3}{2} < \gamma < \infty$, $1 < \alpha \leq \infty$, and $\frac{2}{\alpha}+\frac{3}{\gamma} \leq 2$, or*

(ii) $u^r \in L^\alpha(0,T;L^\gamma(\Gamma_\delta))$ *where α and γ satisfy $3 < \gamma < \infty$, $2 < \alpha \leq \infty$, and $\frac{2}{\alpha}+\frac{3}{\gamma} \leq 1$.*

Then u is smooth in $\mathbb{R}^3 \times (0,T)$.

Remark 3. Comparing with the result in [18] on the regularity criterion in terms of single component of velocity for the general case(without assumption of any symmetry of solutions), we find that in Theorem 4 the integrability assumption for the single component is much weaker, and optimal in the sense of scaling dimension of the norms.

A result similar to Theorem 4 (ii) was obtained independently at about the same time by Neustupa et al., see [19].

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Boundary singular sets for Stokes equations

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Abstract

Removable singular sets on the boundary for Stokes equations are considered. We prove that the capacity function lies in a suitable fractional Sobolev space. This implies that singular sets with small capacity are removable to fractional Sobolev solutions. Then we prove that Poisson kernel to Stokes equations maps the capacity measures on the boundary to suitable fractional Sobolev spaces in domain. Thus we find a criterion for the capacity of removable singular sets and the regularity of the Sobolev solutions.

1 Introduction

In this note we study singular sets on the boundary for Stokes equations. Mainly we are concerned about the size of singular sets and removability. The relation between the size of interior singular set and removability is relatively well understood for Laplace equations. In particular We refer to the excellent book by Carleson(see [1]). Also the behaviour of solutions to Stokes and Navier-Stokes equations near isolated singular point are studied by several authors(see [3] and [6]). When the singular sets are large, the languages of capacity and Hausdorff measure are most convenient. Indeed Shapiro studied the removability of interior singular sets for Navier-Stokes equations in terms of Newtonian capacity when the space dimension is three.

On the other hand Chesnokov and Egorov studied boundary singular sets for linear partial differential equations. There the regularity assumptions of solutions are described in terms of fractional Sobolev space $W^{s,2}$ which are Hilbert spaces. He found some optimal relations between the capacities of singular sets and regularity assumptions on the solutions.

Suppose that $A \subset R^n$. The usual Riesz capacity $Cap_p(A : R^n)$ is equal to $\sup \mu(A)$, where the upper bound is taken over all positive Borel measures μ with

their supports in A for which $\int |x-y|^{n-p} d\mu(y) \leq 1$. Define a ball by $B_r(x_0) = \{x : |x - x_0| < r\}$. Then the Hausdorff measure is defined in the following way:

$$H^\alpha(A) = \lim_{\varepsilon \to 0} \inf \sum_{i=1}^{k} |r_i|^\alpha,$$

where the infimum is taken over all the finite union of the balls $B_{r_i}(x_i), r_i \leq \varepsilon$ which cover A.

Let $\Omega \subset R^{n+1}, n \geq 1$ be an open bounded domain with smooth boundary. We define the fractional Sobolev space $W^{s,q}(\Omega), 1 > s > 0, q \geq 1$ by the set of all functions v in $L^q(\Omega)$ such that

$$\sup_{x,y\in\Omega} \int\int \frac{|v(x)-v(y)|^q}{|x-y|^{n+1+sq}} dxdy < \infty.$$

Here we find the removability criterion in terms of capacity of the singular sets on the boundary and regularity assumptions on the solutions for Stokes equations. One important ingredient is that the capacity function lies in a suitable fractional Sobolev space(see Lemma 2.2). Moreover we prove that the Poisson kernel maps capacity measure to suitable fractional Sobolev space. The following Theorem is our main result.

Theorem 1.1. *Suppose $u \in W^{s,m}(\Omega), p \in W^{s-1,m}(\Omega), 1 \leq m \leq 2, 0 < s < \frac{1}{m}$ is a solution to Stokes equations and $A \subset \partial\Omega$. Let*

$$\theta = \frac{m}{m-1}(1-s) - 1.$$

Then if $Cap_\gamma(A : R^n) = 0, \gamma > \theta$, then A is removable. On the other hand if $Cap_\gamma(A : R^n) > 0, \gamma < \theta$, then A is irremovable.

We denote a constant depending only on exterior data by c and double indices mean summation.

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2 Boundary singular sets for Stokes equations

We let $(u,p) : R^{n+1} \to R^{n+1} \times R$ be a solution to Stokes equations

$$\Delta u - \nabla p = 0, \qquad \nabla \cdot u = 0$$

in a bounded domain $\Omega \subset R^{n+1}$. For simplicity we assume that $\Omega \subset R^n \times R^+$ and $\Gamma \subset \partial\Omega$ is an open subset of $R^n \times \{0\}$.

From integration by parts we have

$$\int_{\Omega}\left[\Delta u^i - \frac{\partial p}{\partial x_i}\right] v^i - u^i\left[\Delta v^i + \frac{\partial q}{\partial x_i}\right] dx - \int p\nabla\cdot v dx$$

$$= \int_{\partial\Omega} T_{ij}(u,p)v^i\nu^j - T^{'}_{ij}(v,q)u^i\nu^j d\sigma_y$$

for all smooth (v, q), where $\nu(y)$ is the outward normal vector at $y \in \partial\Omega$ and

$$T_{ij}(u,p) = -\delta_{ij}p + \frac{\partial u^i}{\partial x_j}$$

and

$$T^{'}_{ij}(v,q) = \delta_{ij}q + \frac{\partial v^i}{\partial x_j}.$$

When (u,p) is a $L^1(\Omega)$ weak solution of Stokes equations, we define the trace u_Γ of u on Γ as a functional in $\mathcal{D}^{'}(\Gamma)$ by the formula

$$(u_\Gamma, \phi) = \int_{\Omega} u^i\left[\Delta v^i + \frac{\partial q}{\partial x_i}\right] dx + \int_{\Omega} p\nabla\cdot v dx,$$

where $\phi \in \mathcal{D}(\Gamma), (v,q) \in \mathcal{D}(\bar{\Omega}), v = 0$ on $\partial\Omega$ and (v,q) satisfies

$$T^{'}_{ij}(v,q)\nu^j = \phi^i$$

for all $i = 1, ..., n, n+1$.

We let $A \subset \Gamma$ be a compact subset and $u_\Gamma = 0$ on $\Gamma \setminus A$ in the sense of trace. We want to know a relation between the size of singular set A and the regularity of u to decide that $u_\Gamma = 0$ on whole Γ. If u_Γ equals to zero across A, we say A is removable. First we consider the case when u lies in Lebesgue integrable spaces and A has a finite Hausdorff dimension.

Theorem 2.1. *We suppose that $u \in L^m(\Omega), m \in [1,\infty)$ and $p \in L^1(\Omega)$. If $H^{\frac{n(m-1)-1}{m}}(A) = 0$, then A is removable.*

remark. *The Poisson kernel for velocity is supported at a single point which has a nonzero 0-dimensional Hausdorff measure. Moreover it lies in weak-$L^{\frac{n+1}{n}}$ space and the trace is not zero.*

proof. For notational simplicity, we denote $t = x_{n+1}$ and $x = (x_1, \cdots, x_n)$. If $m < \frac{n+1}{n}$, then $\frac{n(m-1)-1}{m} < 0$ and A is empty. Thus the theorem holds trivially. Suppose that $u \in L^m(\Omega)$ and $p \in L^1(\Omega), m \geq \frac{n+1}{n}$. Since A has zero $\frac{n(m-1)-1}{m}$-dimensional measure, for given ε there exist $\{B_{r_k}(x^k) : |r_k| \leq \varepsilon, x^k \in A\}$ such that $A \subset \cup B_{r_k}(x^k)$ and

$$\lim_{\varepsilon\to 0}\sum_k (r^k)^{\frac{n(m-1)-1}{m}} = 0.$$

Moreover we can assume that for each fixed ε, $\frac{\varepsilon}{2} \leq r_k \leq \varepsilon$ for all k. We let $\alpha^k(x,t)$ be a shielding function of $B_{r_k}(x^k) \times \{0\}$ such that $\alpha^k(x,t) = 0$ in $B_{r_k}(x^k) \times (0, r_k)$, $\alpha^k(x,t) = 1$ in $R^n \times R^+ \backslash B_{2r_k}(x^k) \times (0, 2r_k)$, $|\nabla \alpha^k| \leq \frac{c}{r_k}$ and $|\nabla^2 \alpha^k| \leq \frac{c}{r_k^2}$. Moreover we can assume that $\{B_{r_k}(x^k)\}$ has finite intersection property, that is, for each x the number of the balls which contain x is bounded by a constant depending only on n. Let $h_\varepsilon(x,t) = 1 - \sum_k \alpha_k(x,t)$. Also we let $g_\varepsilon : R^+ \cup \{0\} \to R$ be a smooth function such that $g_\varepsilon = 1$ in $[0, \varepsilon)$ and $g_\varepsilon = 0$ in $[2\varepsilon, \infty)$ and $|g_\varepsilon'| \leq \frac{c}{\varepsilon}$.

Suppose that $\phi \in C_0^\infty(\Gamma : R^{n+1})$ and $v = t\phi g_\varepsilon h_\varepsilon$. Moreover we define $q = 0$. Then, we find that

$$\frac{\partial v}{\partial \nu}(x,0) = -\frac{\partial}{\partial t}\left[t\phi g_\varepsilon h_\varepsilon\right] = -\phi h_\varepsilon(x)$$

for outward unit normal vector ν and since $v(x,0) = 0$ for all $(x,0) \in \Gamma$,

$$\frac{\partial v}{\partial \tau} = 0$$

for all tangent vector τ at each $(x,0) \in \Gamma$. Hence from integration by parts we find that

$$\begin{aligned}
&\int_\Omega u^i(\Delta v^i + \frac{\partial q}{\partial x_i}) dxdt + \int_\Omega p \nabla \cdot v dxdt \\
&= \int_\Omega u^i \Delta\left(t\phi^i g_\varepsilon h_\varepsilon\right) dxdt + \int_\Omega p \Big[t g_\varepsilon \nabla \cdot (\phi h_\varepsilon) \\
&\qquad + t\phi^{n+1} g_\varepsilon' h_\varepsilon + \phi^{n+1} g_\varepsilon h_\varepsilon \Big] dxdt \\
&= \int_\Gamma T_{ij}'(v,0) u^i \nu_j dx.
\end{aligned}$$

Observe that u is smooth in $\overline{\Omega} \setminus \cup B_{r_k}(x_k)$ and u is well defined on $\Gamma \setminus \cup B_{r_k}(x_k)$. Since $g_\varepsilon = 1$ on Γ and $\frac{\partial v}{\partial \nu}(x,0) = -\phi h_\varepsilon(x,0)$, we have

$$\begin{aligned}
-\int_\Gamma u^i \phi^i h_\varepsilon dx &= \int_\Omega u^i \Delta\left(t\phi^i g_\varepsilon h_\varepsilon\right) dxdt \\
&\quad + \int_\Omega p \Big(t g_\varepsilon \nabla \cdot (\phi h_\varepsilon) + \phi^{n+1} g_\varepsilon h_\varepsilon \\
&\qquad + t\phi^{n+1} g_\varepsilon' h_\varepsilon \Big) dxdt.
\end{aligned}$$

Since we are assuming $p \in L^1(\Omega)$, we see that

$$\left| \int_\Omega p \left(t \nabla \cdot (\phi g_\varepsilon h_\varepsilon) + \phi^{n+1} g_\varepsilon h_\varepsilon + t\phi^{n+1} g_\varepsilon' h_\varepsilon \right) dxdt \right| \to 0$$

as ε goes to zero. On the other hand we have

$$\int_\Omega u^i \Delta\left(t\phi^i g_\varepsilon h_\varepsilon\right) dxdt$$

$$= \int_\Omega tu^i(\Delta\phi^i)g_\varepsilon h_\varepsilon dxdt + \int_\Omega tu^i\phi^i g_\varepsilon \Delta h_\varepsilon dxdt$$

$$+ \int_\Omega tu^i g_\varepsilon \nabla\phi^i \cdot \nabla h_\varepsilon dxdt + \int_\Omega tg_\varepsilon'' u^i\phi^i h_\varepsilon dxdt$$

$$+ \int_\Omega g_\varepsilon' u^i\phi^i h_\varepsilon dxdt$$

$$= I_\varepsilon + II_\varepsilon + III_\varepsilon + IV_\varepsilon + V_\varepsilon.$$

Since $||\Delta\phi||_\infty < \infty$ and $u \in L^m$,

$$|I_\varepsilon| \le ||\Delta\phi||_\infty \left(\int_\Omega |u|^m dxdt\right)^{\frac{1}{m}} \left(\int_\Omega (tg_\varepsilon h_\varepsilon)^{\frac{m}{m-1}} dxdt\right)^{\frac{m-1}{m}}$$

and

$$\lim_{\varepsilon\to 0} |I_\varepsilon| = 0.$$

Now we recall that $||\Delta\alpha^k||_\infty \le \frac{c}{r_k^2}$. Thus we obtain that

$$|II_\varepsilon| \le ||u||_m \left[\sum_k \int_\Omega |tg_\varepsilon \Delta\alpha^k|^{\frac{m}{m-1}} dxdt\right]^{\frac{m-1}{m}}$$

$$\le ||u||_m \left[\int_\Gamma |\Delta \sum_k \alpha^k|^{\frac{m}{m-1}} dx\right]^{\frac{m-1}{m}} \left[\int_0^{2\varepsilon} g_\varepsilon^{\frac{m}{m-1}} t^{\frac{m}{m-1}} dt\right]^{\frac{m-1}{m}}$$

$$\le ||u||_m \sum_k \left[\int_\Gamma |\Delta\alpha^k|^{\frac{m}{m-1}} dx\right]^{\frac{m-1}{m}} \left[\int_0^{2r_k} g_\varepsilon^{\frac{m}{m-1}} t^{\frac{m}{m-1}} dt\right]^{\frac{m-1}{m}}$$

$$\le c \sum_k r_k^{\frac{n(m-1)-1}{m}}$$

for some c. Since $H^{\frac{n(m-1)-1}{m}}(A) = 0$, $\lim_{\varepsilon\to 0} |II_\varepsilon| = 0$. Similarly

$$\lim_{\varepsilon\to 0} |III_\varepsilon| = \lim_{\varepsilon\to 0} |IV_\varepsilon| = \lim_{\varepsilon\to 0} |V_\varepsilon| = 0.$$

This completes the proof.

Now we consider the case when u lies in a fractional Sobolev space $W^{s,m}$. The following lemma for capacity function is essential.

Lemma 2.2. *Suppose that μ is a nonnegative Borel measure defined on R^n and $n > m > 0$. If $\mu(R^n) < \infty$ and its Riesz potential satisfies*

$$R_m(\mu)(x) = \int_{R^n} \frac{d\mu(y)}{|x-y|^{n-m}} \leq c,$$

then

$$R_m(\mu)(x) \in W^{\alpha,r}_{loc}(R^n)$$

for all $1 \leq r \leq 2$ and $\alpha r < m$.

proof. We show that $R_{m-\alpha}(\mu) \in L^r_{loc}(R^n)$, if $1 \leq r \leq 2$ and $\alpha r < m$. Let $D \subset R^n$ be a bounded set. ¿From Fubini theorem we have

$$\int_D \left| \int_{R^n} \frac{d\mu(y)}{|x-y|^{n-m+\alpha}} \right|^r dx$$

$$= \int_{R^n} d\mu(y) \int_D \frac{1}{|x-y|^{n-m+\alpha}} \left[\int_{R^n} \frac{d\mu(z)}{|x-z|^{n-m+\alpha}} \right]^{r-1} dx.$$

On the other hand

$$\int_D \frac{1}{|x-y|^{n-m+\alpha}} \left[\int_{R^n} \frac{d\mu(z)}{|x-z|^{n-m+\alpha}} \right]^{r-1} dx$$

$$\leq \left[\int_D \frac{1}{|x-y|^{(n-m+\alpha)\frac{1-a}{2-r}}} dx \right]^{2-r} \cdot$$

$$\left[\int_D \frac{1}{|x-y|^{(n-m+\alpha)\frac{a}{r-1}}} \int_{R^n} \frac{d\mu(z)}{|x-z|^{n-m+\alpha}} dx \right]^{r-1}$$

$$\leq \left[\int_D \frac{dx}{|x-y|^{(n-m+\alpha)\frac{1-a}{2-r}}} \right]^{2-r} \left[\int_{R^n} \frac{d\mu(z)}{|y-z|^{(n-m+\alpha)(1+\frac{a}{r-1})-n}} \right]^{r-1}.$$

Now we choose $a = \frac{(n-\alpha)(r-1)}{n-m+\alpha}$. Then since $\alpha r < m$, we find that $\frac{(n-m+\alpha)(1-a)}{2-r} < n$ and $(n-m+\alpha)(1+\frac{a}{r-1}) - n = n - m$. Since we are assuming

$$\int_{R^n} \frac{d\mu(z)}{|x-z|^{n-m}} \leq c,$$

we get

$$\int_{R^n} \frac{1}{|x-y|^{n-m+\alpha}} \left[\int_{R^n} \frac{d\mu(z)}{|x-z|^{n-m+\alpha}} \right]^{r-1} dx \leq c.$$

Thus $f(x) = R_{m-\alpha}(\mu)(x) \in L^r$ and

$$R_m(\mu)(x) = R_\alpha(f)(x).$$

Therefore from the definition of fractional Sobolev space we get $R_m(\mu) \in W^{\alpha,r}$.

From Lemma 2.2 we find that if $Cap_m(A : R^n) = 0$, then for each $\varepsilon > 0$ there exists h_ε such that $h_\varepsilon = 1$ on A and

$$\lim_{\varepsilon \to 0} ||h_\varepsilon||_{\alpha,r} = 0$$

for each $1 \leq r \leq 2, \alpha r < m$.

The following trace lemma is relatively well known. We let $v \in \mathcal{D}(\Omega)$ and denote its boundary value by $\gamma_0 v$ and its normal derivative by $\gamma_1 v = \frac{\partial v}{\partial \nu}$ at $\Gamma \subset \partial\Omega$.

Lemma 2.3. *Let $m \geq 1$ and $s \geq 0$ be two real numbers such that $1 \leq s \leq k+1, s - \frac{1}{m} = l + \sigma$, where $l \geq 1$ is an integer and $0 < \sigma < 1$. Then, the map*

$$v \to (\gamma_0 v, \gamma_1 u)$$

defined on $\mathcal{D}(\Omega)$ has a unique linear continuous extension as an operator from $W^{s,m}(\Omega)$ onto $W^{s-\frac{1}{m},m}(\partial\Omega) \times W^{s-1-\frac{1}{m},m}(\partial\Omega)$. Moreover

$$||f||_{s-\frac{1}{m},m,\partial\Omega} \leq \inf_{v \in W^{s,m}(\Omega), \gamma_0 v = f} ||v||_{s,m,\Omega}.$$

The following theorem shows a relation between the Sobolev space and the size of singular sets on the boundary.

Theorem 2.4. *Suppose $u \in W^{s,m}(\Omega), p \in W^{s-1,m}(\Omega), 1 \leq m \leq 2, 0 \leq s < \frac{1}{m}$ and $Cap_\gamma(A : R^n) = 0$ for some*

$$\gamma > \theta = \frac{m}{m-1}(1-s) - 1.$$

Then $u_\Gamma = 0$ on Γ in the sense of trace.

proof. From Lemma 2.2 and the assumption that $Cap_\gamma(A) = 0$, we can find a shielding function $g_\varepsilon \in C_0^\infty(R^n)$ such that $g_\varepsilon = 1$ in a neighborhood of A and

$$||g_\varepsilon||_{\alpha,r,R^n} \leq c\varepsilon$$

for all $1 \leq r < \frac{\gamma}{\alpha}$. Then, set $h_\varepsilon = 1 - g_\varepsilon$. Now we fix $\phi \in C_0^\infty(\Gamma)$. From the trace lemma there exists v^ε such that $\gamma_0 v^\varepsilon = 0$ and $\gamma_1 v^\varepsilon = h_\varepsilon \phi$ with

$$||v^\varepsilon||_{2-s,\frac{m}{m-1},\Omega} \leq c||h_\varepsilon \phi||_{1-s-\frac{m-1}{m},\frac{m}{m-1},\Omega}.$$

Now we choose $r = \frac{m}{m-1}$ and $\alpha = 1 - s - \frac{m-1}{m}$. Thus

$$||h_\varepsilon \phi||_{1-s-\frac{m-1}{m},\frac{m}{m-1},R^n} = ||h_\varepsilon \phi||_{\alpha,r,R^n} \leq c\varepsilon.$$

Note that $\alpha r = (1 - s - \frac{m-1}{m})\frac{m}{m-1} = \theta < \gamma$. Now we take $q = 0$ and the above v^ε in the Green's identity. Thus we obtain that

$$\int_\Gamma u^i \phi^i h_\varepsilon dx = \int_\Omega u^i \left(\Delta(v^\varepsilon)^i + \frac{\partial q}{\partial x_i}\right) dx dx_{n+1} + \int_\Omega p\nabla \cdot v^\varepsilon dx dx_{n+1}$$

$$\leq ||u||_{s,m,\Omega}||\Delta v^\varepsilon||_{-s,\frac{m}{m-1},\Omega} + ||p||_{s-1,m,\Omega}||\nabla v^\varepsilon||_{1-s,\frac{m}{m-1},\Omega}$$

$$\leq c\left(||u||_{s,m,\Omega} + ||p||_{s-1,m,\Omega}\right)||v^\varepsilon||_{2-s,\frac{m}{m-1},\Omega}$$

$$\leq \left(||u||_{s,m,\Omega}|| + +||p||_{s-1,m,\Omega}\right)||h_\varepsilon \phi||_{1-s-\frac{m-1}{m},\frac{m}{m-1},R^n}$$

$$\leq c||h_\varepsilon||_{\alpha,r,R^n} \leq c\varepsilon.$$

Therefore sending ε to zero we conclude that

$$\int_\Gamma u^i \phi^i dx = 0$$

for all $\phi \in C_0^\infty(\Gamma)$.

Now we consider the converse of Theorem 2.5. The Poisson type kernel for upper half plane is necessary. Let (G, r) be the Green's tensor for upper half plane(see section 2.2 in [8]). We define the stress tensor on the boundary $x_{n+1} = 0$ by

$$T'_{ij}(G^k, r^k)(x,\xi) = \delta_{ij} r^k(x,\xi) + \left(\frac{\partial G_i^k}{\partial x_j} + \frac{\partial G_j^k}{\partial x_i}\right)(x,\xi).$$

Observe that

$$|T'(G^{n+1}, r^{n+1})(x,\xi)| \leq \frac{c\xi_{n+1}}{|x-\xi|^{n+1}}$$

on $x_{n+1} = 0$(see Theorem 2.2.1 in [8]).

Theorem 2.5. *Let $1 \leq m \leq 2$ and $0 < s < \frac{1}{m}$. Suppose that A is a compact subset of Γ and*

$$0 < Cap_\gamma(A) < \infty, \quad 0 < \gamma < \theta = \frac{m}{m-1}(1-s) - 1.$$

Then there is a solution to Stokes equations in $W^{s,m}(R^n \times R^+)$ with nonzero boundary data supported on A in the sense of trace.

proof. There is a nonzero Borel measure μ supported on A such that

$$\int_{R^n} \frac{d|\mu|(y)}{|x-y|^{n-\gamma}} \leq c,$$

and

$$|\mu|(A) < \infty.$$

We define that

$$u^k(x,t) = -\int_\Gamma T'_{i(n+1)}(G^k, r^k)(x-z,t)d\mu^i(z).$$

First we prove that $u \in L^m(\Omega)$. Set $X = (x,t)$ and $Z = (z,0)$. Also define that

$$K^i_k(X-Z) = T'_{i(n+1)}(G^k, r^k)(x-z,t)$$

Then, from a direct calculation we have

$$\int |u(X)|^m dX \le c\int \left[\int |K(X-Z)|d|\mu|(Z)\right]^m dX$$

$$= \int\int |K(X-Z_1)|d|\mu|(Z_1)\left[\int |K(X-Z)|d|\mu|(Z)\right]^{m-1} dX$$

$$\le \left[\int\int |K(X-Z_1)|dXd|\mu|(Z_1)\right]^{2-m}\cdot$$

$$\left[\int\int\int |K(X-Z_1)K(X-Z)|dXd|\mu|(Z_1))d|\mu|(Z)\right]^{m-1}.$$

Recall that

$$|K(X-Z)| \le c|X-Z|^n.$$

Thus we find that

$$\int |K(X-Z)|\,dX \le c$$

for some c independent of Z. On the other hand we let $\tilde{X} = \frac{X}{|Z_1-Z|}$ and $\tilde{E} = \frac{Z_1-Z}{|Z_1-Z|}$. Then

$$\int |K(X-Z_1)K(X-Z)|dX \le \frac{1}{|\tilde{E}|^{n-1}}\int |K(\tilde{X}-\tilde{Z}_1)K(\tilde{X}-\tilde{Z})|d\tilde{X}$$

$$\le c\frac{1}{|Z_1-Z|^{n-1}}$$

for some c. Note that $0 \le \theta \le 1$. Since $n-1 \le n-\theta$, we find that

$$\int\int |K(X-Z_1)K(X-Z)|dXd|\mu|(Z_1) \le c$$

and $u \in L^m(\Omega)$.

Now we need to estimate the fractional norm

$$\int\int \frac{|u(X)-u(Y)|^m}{|X-Y|^{n+1+sm}}dXdY.$$

We take a small number δ which is fixed later. We let

$$a = (2-m)(n+2) - \delta \quad \text{and} \quad b = n(m-1) + 2m + ms - 3 + \delta$$

Then, $a + b = n + 1 + sm$. As in the case of L^m norm estimate we have

$$\int\int \frac{1}{|X-Y|^{n+1+sm}} \left| \int K(X-Z) - K(Y-Z) d\mu(Z) \right|^m dXdY$$

$$\leq \left[\int\int\int \frac{|K(X-Z) - K(Y-Z)|}{|X-Y|^{\frac{a}{2-m}}} dXdYd|\mu|(Z) \right]^{2-m} \cdot$$

$$\left[\int\int\int\int \frac{|K(X-Z) - K(Y-Z)||K(X-Z_1) - K(Y-Z_1)|}{|X-Y|^{\frac{b}{m-1}}} \right.$$

$$\left. dXdYd|\mu|(Z)d|\mu|(Z_1) \right]^{m-1}$$

If we set $W = X - Y$, then we can write that

$$\int\int \frac{|K(X-Z) - K(Y-Z)|}{|X-Y|^{\frac{a}{2-m}}} dXdY$$

$$= \int\int \frac{|K(W+Y-Z) - K(Y-Z)|}{|W|^{\frac{a}{2-m}}} dWdY$$

$$\leq c \int \frac{1}{|Y-Z|^{n+1-\frac{\delta}{2-m}}} dY.$$

Since $\delta > 0$, we have

$$\int\int\int \frac{|K(X-Z) - K(Y-Z)|}{|X-Y|^{n+1+sm}} dXdYd|\mu|(Z) \leq c.$$

Now we observe that

$$\frac{b}{m-1} = n + 2 + \frac{m-1}{m}s - \frac{1}{m-1} + \frac{\delta}{m-1} = n + 2 - \theta + \frac{\delta}{m-1}.$$

Hence we obtain that

$$\int\int \frac{|K(X-Z) - K(Y-Z)||K(X-Z_1) - K(Y-Z_1)|}{|X-Y|^{\frac{b}{m-1}}} dXdY$$

$$\leq \frac{c}{|Z-Z_1|^{n-\theta+\frac{\delta}{m-1}}}$$

for some c. Now we choose $\delta = (m-1)(\theta - \gamma) > 0$. Thus we prove that

$$\int\int\int\int \frac{|K(X-Z) - K(Y-Z)||K(X-Z_1) - K(Y-Z_1)|}{|X-Y|^{n+1+sm}} dXdYd|\mu|(Z)d|\mu|(Z_1)$$

$$\leq c \int\int \frac{1}{|Z - Z_1|^{n-\gamma}} d|\mu|(Z) d|\mu|(Z_1) \leq c$$

and this proves that $u \in W^{s,m}$.

The measure μ is nonzero, so there exists a function $\psi \in \mathcal{D}(\Gamma)$ such that $(\mu, \psi) \neq 0$. We let $v \in \mathcal{D}(\Omega)$ satisfy $v = 0$ and $\frac{\partial v}{\partial \nu} = \psi$ on Γ. Also considering the pressure potential, we can find the corresponding pressure p to u such that

$$0 \neq (\mu, \psi) = \int_\Omega u^i \Delta v^i dx + \int_\Omega p \nabla \cdot v dx$$

and this proves that $u \neq 0$ in the sense of trace.

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Feedback stabilization for the 2D Navier-Stokes equations

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Abstract

For $2D$ Navier–Stokes equations defined in a bounded domain Ω we study stabilization of solution near a given steady-state flow $\hat{v}(x)$ by means of feedback control defined on a part Γ of boundary $\partial\Omega$. New mathematical formalization of feedback notion is proposed. In the case of linearized Navier–Stokes equations special construction of a feedback control is proposed such that the control reacts on unpredictable fluctuations of fluid velocity suppressing them.The cases of discrete in time and continuous in time unpredictable fluctuations are regarded.

1 Introduction

We study here stability problem for two-dimensional (2D) Navier—Stokes equations defined in a bounded domain $\Omega \subset \mathbb{R}^2$. These equations are controlled by Dirichlet boundary condition for velocity vector field. Let $(\hat{v}(x), \nabla\hat{p}(x))$, $x \in \Omega$ be a steady-state solution for Navier—Stokes equations. Suppose that $\hat{v}$ is an unstable singular point for the dynamic system generated by evolutionary Navier-Stokes system supplied with zero condition $v|_{\partial\Omega} = 0$ on the boundary $\partial\Omega$ of Ω.

The stabilization problem is as follows: Given $\sigma > 0$ and initial condition $v_0(x)$ for evolutionary Navier-Stokes system (v_0 is placed in a small H^1–neighbourhood of $\hat{v}$), find Dirichlet boundary condition (control) $u(t, x')$ defined on $\mathbb{R}_+ \times \partial\Omega$ such that the solution $v(t, x)$ of obtained boundary value problem satisfies inequality

$$\|v(t,\cdot) - \hat{v}(\cdot)\|_{H^1(\Omega)} \leqslant ce^{-\sigma t} \text{ as } t \to \infty. \tag{1.1}$$

Actually, the stabilization result in such formulation follows immediately from exact controllability result of [FE],[F1]. But below we impose on desired control

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very important additional property: u should be feedback control, i.e. u must react on unpredictable fluctuations of v suppressing them.

Problem of stabilization by boundary feedback control was studied earlier mostly for hyperbolic equations and related to them systems (see, for instance, [Li],[Lag],[C]). Some results, connected with Burgers equation was obtained also ([BK]). (Of course, we do not pretend on completeness of references.)

In [F2] new mathematical formalization of feedback property was proposed and stabilization problem for quasilinear parabolic equation with feedback boundary control was solved. Here we get solution for problem of stabilization by boundary feedback control for two-dimensional Navier-Stokes system when a control is concentrated on a part of boundary. Complete proof of this result will be exposed in [F3].

Note that the main part of [F2], [F3] is devoted to study stabilization problem when unpredictable fluctuations do not arise. Only in [F2] we got some preliminary result on suppression of unpredictable fluctuations imposing on these fluctuations very strong conditions. In this paper we obtain new result to this direction. More exactly, in Section 3.3 below we propose new construction of feedback control which made possible to get rid of conditions on unpredictable fluctuations imposed in [F2]. Moreover we think that, in general, the point of view on suppressing of fluctuations proposed here is more adequate to reality than the point of view in [F2]. Here we consider the cases of discrete in time and continuous in time unpredictable fluctuations.

2 Ozeen equations

We begin investigation of stabilization problem from the case of linearized Navier-Stokes equations, i.e. from the Ozeen equations.

2 Formulation of the problem.

Let $\Omega \subset \mathbb{R}^2$, $\partial\Omega \in C^\infty$, $Q = \mathbb{R}_+ \times \Omega$. We consider the Ozeen equations:

$$\partial_t v(t,x) - \Delta v + (a(x), \nabla)v + (v, \nabla)a + \nabla p(t,x) = 0 \tag{2.1}$$

$$\operatorname{div} v(t,x) = 0 \tag{2.2}$$

with initial condition

$$v(t,x)|_{t=0} = v_0(x). \tag{2.3}$$

Here $(t,x) = (t, x_1, x_2) \in Q$, $v(t,x) = (v_1, v_2)$ is a velocity of fluid flow, $\nabla p(t,x)$ is a pressure's gradient, $a(x) = (a_1(x), a_2(x))$ is a given solenoidal vector field ($\operatorname{div} a = 0$).

We suppose that the boundary $\partial\Omega$ of Ω is decomposed on two parts:

$$\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_0, \quad \Gamma \neq \emptyset$$

where Γ, Γ_0 are open sets (in topology of $\partial\Omega$). Here, as usual, the line above means the closer of a set. We define $\Sigma = \mathbb{R}_+ \times \Gamma, \Sigma_0 = \mathbb{R}_+ \times \Gamma_0$, and we set:

$$v|_{\Sigma_0} = 0, \quad v|_\Sigma = u \tag{2.4}$$

where u is a control, concentrated on Σ.

Let a magnitude $\sigma > 0$ be given. The problem of stabilization with the rate σ of a solution to problem (2.1)–(2.4) is to construct a control u defined on Σ such that the solution $v(t,x)$ of boundary value problem (2.1)–(2.4) satisfies:

$$\|v(t,x)\|^2_{L_2(\Omega)} \leqslant ce^{-\sigma t} \tag{2.5}$$

where $c > 0$ depends on v_0, σ and Γ_0. Moreover we require that this control u satisfies the feedback property in the meaning indicatad below.

Let us give the exact formulation of this feedback property. Let $\omega \subset \mathbb{R}^2$ be a bounded domain such that $\Omega \cap \omega = \emptyset$, $\overline{\Omega} \cap \overline{\omega} = \overline{\Gamma}$. We set

$$G = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega}) \tag{2.6}$$

(the denotion $\mathrm{Int}\, A$ means, as always, the interior of the set A). We suppose that $\partial G \in C^\alpha$ and in all points except $\overline{\Gamma} \setminus \Gamma \equiv \partial\Gamma$ it possesses the C^∞ smoothness. The exact conditions on α will be imposed below in an appropriate place.

We extend problem (2.1)–(2.3) from Ω to G. Let us assume that

$$a(x) \in V^2(G) \cap \left(H_0^1(G)\right)^2 \tag{2.7}$$

where, as usually, $H^k(G)$, $k \in \mathbb{N}$, is the Sobolev space of scalar functions, defined and square integrable on G together with all its derivatives up to order k and $(H^k(G))^2$ is the analogous Sobolev space of vector fields. Besides, $H_0^1(G) = \{f(x) \in H^1(G) : f(x)|_{x\in\partial G} = 0\}$ and

$$V^k(\Omega) = \left\{v(x) = (v_1, v_2) \in (H^k(\Omega))^2 : \quad \mathrm{div}\, v = 0\right\}$$

The extension of (2.1)–(2.3) from Ω to G can be written as follows:

$$\partial_t w(t,x) - \Delta w + (a(x), \nabla)w + (w, \nabla)a + \nabla p(t,x) = 0 \tag{2.8}$$

$$\mathrm{div}\, w(t,x) = 0 \tag{2.9}$$

$$w(t,x)|_{t=0} = w_0(x) \tag{2.10}$$

Moreover we impose on w the zero Dirichlet boundary condition

$$w|_S = 0, \tag{2.11}$$

where $S = \mathbb{R}_+ \times \partial G$. Note that actually w_0 from (2.10) will be some special extension of $v_0(x)$ from (2.3).

For vector fields defined on G we denote by γ_Ω the operator of restriction on Ω and by γ_Γ we denote the operator of restriction on Γ:

$$\gamma_\Omega : V^k(G) \longrightarrow V^k(\Omega), \quad \gamma_\Gamma : V^k(G) \longrightarrow V^{k-1/2}(\Gamma), \ k \geqslant 0$$

Evidently, these operators are bounded.

Definition 2.1 A control $u(t,x)$ in (2.1)–(2.4) is called feedback if

$$v(t,\cdot) = \gamma_\Omega w(t,\cdot), \quad u(t,\cdot) = \gamma_\Gamma w(t,\cdot) \qquad \forall t \geqslant 0 \tag{2.12}$$

where $(v(t,\cdot), u(t,\cdot))$ is the solution of stabilization problem (2.1)–(2.4) and $w(t,\cdot)$ is the solution of boundary value problem (2.8)–(2.11).

Below we prove that for given $\sigma > 0, v_0$ the problem (2.1)– (2.4) can be stabilized with help of feedback control in the meaning of Definition 2.1.

2 Preliminaries.

Let G be domain (2.6) and

$$V_0^0(G) = \{v(x) \in V^0(G) : v \cdot \nu|_{\partial\Omega} = 0\} \tag{2.13}$$

where $\nu(x)$ is the vector-field of outer normals to ∂G. Denote by

$$\pi : (L_2(G))^2 \longrightarrow V_0^0(G) \tag{2.14}$$

the operator of orthogonal projection. We consider the Ozeen steady state operator

$$Av \equiv -\pi\Delta v + \pi[(a(x),\nabla)v + (v,\nabla)a] : V_0^0(G) \longrightarrow V_0^0(G) \tag{2.15}$$

where $a(x)$ is vector-field (2.7). This operator is closed and has the domain:

$$\mathcal{D}(A) = V^2(G) \cap (H_0^1(G))^2 \tag{2.16}$$

which is dense in $V_0^0(G)$.

Asumming that spaces in (2.14), (2.16) are complex we denote by $\rho(A)$ the resolvent set of operator A, i.e. the set of $\lambda \in \mathbb{C}$ such that the resolvent operator

$$R(\lambda, A) \equiv (\lambda I - A)^{-1} : V_0^0(G) \longrightarrow V_0^0(G) \tag{2.17}$$

is defined and continuous. Here I is identity operator. Denote by $\Sigma(A) \equiv \mathbb{C}^1 \setminus \rho(A)$ the spectrum of operator A.

As well-known, Ozeen operator (2.15) is sectorial, i.e. there exist $\varphi \in (0, \pi/2)$, $M \geqslant 1, a \in \mathbb{R}$ such that

$$S_{a,\varphi} = \{\lambda \in \mathbb{C} : \quad \varphi \leqslant |\arg(\lambda - a)| \leqslant \pi, \quad \lambda \neq a\} \subset \rho(A) \tag{2.18}$$

and $\|(\lambda I - A)^{-1}\| \leqslant M/|\lambda - a|, \quad \forall \lambda \in S_{a,\varphi}$. Besides, for $\lambda \in \rho(A)$ resolvent (2.17) is a compact operator, and the spectrum $\Sigma(A)$ consists of a discrete set of points.

We decompose the resolvent $R(\lambda, -A)$ in a neighbourhood of $-\lambda_j \in \Sigma(-A)$:

$$R(\lambda, -A) = \sum_{k=-m}^{\infty} (\lambda + \lambda_j)^k R_k, \quad R_k = (2\pi i)^{-1} \int\limits_{|\lambda+\lambda_j|=\varepsilon} (\lambda + \lambda_j)^{-k-1} R(\lambda, -A) d\lambda \tag{2.19}$$

Note that $m < \infty$.

Let us consider the adjoint operator A^* to Ozeen operator (2.15). Evidently, A^* is a closed operator with domain $\mathcal{D}(A^*) = V^2(G) \cap (H_0^1(G))^2$. Moreover, A^* is sectorial with a compact resolvent and

$$\rho(A^*) = \overline{\rho(A)} \quad \text{and} \quad R(\lambda, A)^* = R(\bar{\lambda}, A^*) \quad \forall\, \lambda \in \rho(A) \tag{2.20}$$

(Here the line above means complex conjugation.) Below we always assume that

$$\text{vector field } a(x) \text{ from (2.7), (2.15) is real valued.} \tag{2.21}$$

That is why we have $\rho(A) = \overline{\rho}(A) = \rho(A^*) = \overline{\rho}(A^*)$.

2 Structure of R_k with $k < 0$

Let $-\lambda_j \in \Sigma(-A)$ be an eingenvalue of $-A$, and $e \neq 0$, $e \in \ker(\lambda_0 I + A)$ be an eingenvector. Vector e_k is called joined vector of order k to e if e_k satisfies:

$$(\lambda_0 I + A)e = 0, \quad e + (\lambda_0 I + A)e_1 = 0, \ldots, \quad e_{k-1} + (\lambda_0 I + A)e_k = 0.$$

We say that $e, e_1, e_2, \ldots$ form a chain of joined vectors. The maximal order m of vectors, joined to e is finite and the number $m + 1$ is called multiplicity of the eigenvector e.

Definition 2.2 The set of eigenvectors and joined vectors

$$e^{(k)}(-\lambda_j), e_1^{(k)}(-\lambda_j), \ldots, e_{m_k}^{(k)}(-\lambda_j) \qquad (k = 1, 2, \ldots, N(-\lambda_j)) \tag{2.22}$$

corresponding to an eigenvalue $-\lambda_j$ is called canonical system if:

i) Vectors $e^{(k)}(-\lambda_j), k = 1, 2, \ldots, N(-\lambda_j)$ form a basis in the space of eigenvectors corresponding to the eigenvalue $-\lambda_j$.

ii) $e^{(1)}(-\lambda_j)$ is an eigenvector with maximal possible multiplicity.

iii) $e^{(k)}(-\lambda_j)$ is an eigenvector which can not be expressed by a linear combination of $e^{(1)}(-\lambda_j), \ldots, e^{(k-1)}(-\lambda_j)$ and multiplicity of $e^{(k)}(-\lambda_j)$ achives a possible maximum.

iy) Vectors (2.22) with fixed k form a complete chain of joined elements.

Besides canonical system (2.22) which corresponds to an eigenvalue $-\lambda_j$ of operator $-A$ we consider a canonical system

$$\varepsilon^{(k)}(-\overline{\lambda}_j), \varepsilon_1^{(k)}(-\overline{\lambda}_j), \ldots, \varepsilon_{m_k}^{(k)}(-\overline{\lambda}_j) \qquad (k = 1, 2, \ldots, N(-\overline{\lambda}_j)) \tag{2.23}$$

that corresponds to the eigenvalue $-\overline{\lambda}_j$ of the ajoint operator $-A^*$. Definition of canonical system (2.23) is absolutely analogous to Definition 2.2 of canonical system (2.22). We define canonical system (2.23) by $E^*(-\overline{\lambda}_j)$.

Theorem 2.1 *Let R_k are operators defined in (2.19). Then*

$$R_{-k}x = 0, \quad \forall k = 1, 2, \ldots, m$$

if and only if

$$\langle x, \varepsilon_l^{(k)}(-\overline{\lambda}_j)\rangle = 0 \qquad \forall \varepsilon_l^{(k)}(-\overline{\lambda}_j) \in E^*(-\overline{\lambda_j})$$

This assertion follows immediately from one result of Keldysh ([K]) on structure of the main part of Laurent serie for $R(\lambda, -A)$. The proof of Theorem 2.1 see in [F3].

2 Holomorphic semigroups.

We regard boundary value problem (2.8)–(2.11) for Ozeen equations written in the form

$$\frac{dw(t)}{dt} + Aw(t) = 0, \quad w|_{t=o} = w_0. \tag{2.24}$$

where A is operator (2.15). Then for each $w_0 \in V_0^0(G)$ the solution $w(t, \cdot)$ of (2.24) is defined by $w(t, \cdot) = e^{-At}w_0$ and

$$e^{-At} = (2\pi i)^{-1} \int_\gamma (\lambda I + A)^{-1} e^{\lambda t} d\lambda, \tag{2.25}$$

where γ is a contour belonging to $\rho(-A)$ such that $\arg\lambda = \pm\theta$ foı$\lambda \in \gamma, |\lambda| \geqslant N$ for certain $\theta \in (\pi/2, \pi)$ and for sufficiently large N. Moreover, γ surrounds $\Sigma(-A)$ from the right. Such contour γ exists, of course, because we can choose γ belonging to set $-S_{a,\varphi}$ from (2.18).

Let $\sigma > 0$ satisfy:

$$\Sigma(-A) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda = -\sigma\} = \emptyset \tag{2.26}$$

The case when there are certain points of $\Sigma(-A)$ placed righter than the line $\{\mathrm{Re}\lambda = -\sigma\}$ will be interesting for us. By γ_σ we denote the continuous contur that is placed in $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leqslant -\sigma\}$ and constructed from an interval of the line $\{\mathrm{Re}\lambda = -\sigma\}$ and from two branches of contour γ that transform to $\{\arg\lambda = \theta\}$ and $\{\arg\lambda = -\theta\}$, $\theta \in (\pi/2, \pi)$ for sufficiently large $|\lambda|$.

We reduce integration over γ in (2.25) to integration over γ_σ and integration around poles $-\lambda_j$ from (2.19) for λ_j satisfying $\mathrm{Re}\lambda_j < \sigma$ After calculation corresponding residues we transform (2.25) to the equality:

$$e^{-At} = (2\pi i)^{-1} \int\limits_{\gamma_{\sigma_0}} (\lambda E + A)^{-1} e^{\lambda t} d\lambda + \sum_{\mathrm{Re}\lambda_j < \sigma} e^{-\lambda_j t} \sum_{n=1}^{m(-\lambda_j)} \frac{t^{n-1}}{(n-1)!} R_{-n}(-\lambda_j). \tag{2.27}$$

We denote

$$V_0^1(G) = \{v \in V^1(G) : v|_{\partial G} = 0\}.$$

Equality (2.27) implies

Theorem 2.2 *Suppose that A is operator* (2.15) *and $\sigma > 0$ satisfies* (2.26). *Then for each $w_0 \in V_0^1(G)$ that satisfies*

$$\langle w_0, \varepsilon_l^{(k)}(-\bar{\lambda}_j) \rangle = 0, \quad l = 0, 1, \dots, m_k,\ k = 1, 2, \dots, N(-\lambda_j),\ \mathrm{Re}(\lambda_j) < \sigma$$

(*here by definition $\varepsilon_0^{(k)}(-\bar{\lambda}_j) = \varepsilon^{(k)}(-\bar{\lambda}_j)$*) *the following inequality holds:*

$$\|e^{At} w_0\|_{V_0^1(G)} \leqslant c e^{-\sigma t} \|w_0\|_{V_0^1(G)} \quad \text{for } t \geqslant 0 \tag{2.28}$$

Proof see in [F2], [F3].

2 On linear independence of $\varepsilon_l^{(k)}(x, -\bar{\lambda}_j)$.

We set some strengthening of well-known result on linear independence of eigenvectors and joint vectors for operator A^* conjugate to Ozeen operator (2.15).

Theorem 2.3 *Consider the set*

$$E_\sigma^* \equiv \bigcup_{\mathrm{Re}\lambda_j < \sigma} E^*(-\bar{\lambda}_j)$$

of canonical systems (2.23) for operator $-A^*$ with σ satisfying (2.26). *Then for arbitrary subdomain $\omega \subset G$ the set $\varepsilon_l^{(k)}(x, -\bar{\lambda}_j) \in E_\sigma^*$ of vector fields regarded for $x \in \omega$ are linear independent.*

The proof of this assertion is given in [F2],[F3].

Impose on canonical systems (2.30) the following condition

$$\varepsilon^{(k)}(-\lambda_j) = \bar{\varepsilon}^{(k)}(-\bar{\lambda}_j); \quad \varepsilon_l^{(k)}(-\bar{\lambda}_j) = \bar{\varepsilon}_l^{k}(-\lambda_j) \tag{2.29}$$

Condition (2.29) can be realized by (2.21).

In virtue of (2.29) canonical system corresponding to real $-\lambda_j$ consists of real valued vector fields. If $\mathrm{Im}\lambda_j \neq 0$, instead of vector fields $\varepsilon^{(k)}(-\bar{\lambda}_j)$, $\bar{\varepsilon}_l^{(k)}(-\lambda_j)$, $l = 0, 1, \dots$, we consider real valued vector fields

$$\mathrm{Re}\varepsilon_l^{(k)}(-\bar{\lambda}_j), \mathrm{Im}\varepsilon_l^{(k)}(-\bar{\lambda}_j), \quad l = 1, \dots, \quad k = 1, 2 \dots \tag{2.30}$$

(with $\varepsilon_0^{(k)}(-\bar{\lambda}_j) = \varepsilon^{(k)}(-\bar{\lambda}_j)$ by definition). We renumber all functions (2.30) with $\mathrm{Re}\lambda_j < \sigma$ (including fields with $\mathrm{Im}\lambda_j = 0$) as follows:

$$\varepsilon_1(x), \dots, \varepsilon_K(x) \tag{2.31}$$

Lemma 2.1 *For an arbitrary subdomain $\omega \subset G$ vector fields* (2.31) *restricted on ω are linear independent over the field $\mathbb{R}$ of real numbers.*

Lemma 2.1 follows easily from Theorem 2.3.

Note that Theorem 2.2 implies immediately the following assertion.

Corollary 2.1 *Assume, that A is operator* (2.18) *and $\sigma > 0$ satisfies* (2.26). *Then for each $w_0 \in V_0^0(G)$ satisfying*

$$\int_G (w_0(x), \varepsilon_j(x))\partial x = 0, \qquad j = 1,..,K$$

with ε_j from (2.31), *inequality* (2.28) *is true.*

3 Stabilization of the Ozeen equation

3 Theorem on extension

The key step in stabilization method that we propose is construction of special extension for initial vector field $v_0(x)$ from (2.3) defined on Ω to a vector field defined on G which we take as initial value $w_0(x)$ in (2.10). First of all we make more precise the conditions imposed on Ω and G. Recall that

$$\partial\Omega = \Gamma \cup \Gamma_0 \cup \partial\Gamma, \quad \partial G \cap \partial\Omega = \Gamma_0 \cup \partial\Gamma \tag{3.1}$$

where Γ, Γ_0 are open subsets of $\partial\Omega$, $\Gamma \neq \emptyset$, and $\partial\Gamma$ is a finite number of points, or $\partial\Gamma = \emptyset$. We suppose that

$$\partial\Omega = \bigcup_{j=1}^{N} \partial\Omega_j \tag{3.2}$$

where $\partial\Omega_j$ are connected components of $\partial\Omega$. We impose

Condition 3.1. For each $j = 1, \dots, N$ the set $\partial\Omega_j \cap \Gamma_0$ is connected or empty.

Condition 3.2. Let $\partial\Omega \in C^\infty$, $\partial G \setminus \partial\Gamma \in C^\infty$ and for each point $P \in \partial\Gamma$ there exist local coordinates (x, y) such that P is origin; $P = (0,0)$, $\{(x,0),\, x \in (0,\varepsilon)\} \subset \Gamma$, $\{(x,0),\, x \in (-\varepsilon, 0)\} \subset \Gamma_0$ where $\varepsilon > 0$ is small enough and

$$\partial G \supset \{(x,y) = (x, x^\alpha),\ x \in (0,\varepsilon)\} \cup \{(x,y) = (x,0),\ x \in (-\varepsilon, 0)\}, \quad \alpha \geqslant 2. \tag{3.3}$$

Since Ω is a given domain where stabilized Ozeen system is determined, and domain ω we choose ourselves, Condition 3.2 is not restrictive.

We introduce the following space

$$V^1(\Omega, \Gamma_0) = \{u(x) \in V^1(\Omega) \,:\, u|_{\partial\Gamma_0} = 0,\ \exists\, v \in V_0^1(G) \text{ that } u = \gamma_\Omega v\} \tag{3.4}$$

where γ_Ω is the operator of restriction on Ω for functions from $V_0^1(G)$. The space $V^1(\Omega, \Gamma_0)$ is supplied with the following norm:

$$\|u\|_{V^1(\Omega,\Gamma_0)} = \inf_L \|Lu\|_{V_0^1(G)}$$

where L runs through all bounded extension operators $L : V^1(\Omega, \Gamma_0) \to V_0^1(G)$.

Remind that for a scalar function $F(x)$ $\text{rot}F(x) = (\partial_{x_2}F(x), -\partial_{x_1}F(x))$ and for vector field $v(x) = (v_1(x), v_2(x))$ $\text{rot}^*v(x) = \partial_1 v_2(x) - \partial_2 v_1(x)$.

We prove now the extension theorem. In the space of real valued vector fields $V_0^1(G)$ we introduce the subspace

$$X_\sigma^1(G) = \{v(x) \in V_0^1(G) : \int_G v(x) \cdot \varepsilon_j(x)\, dx = 0, \quad j = 1, \dots, K\} \tag{3.5}$$

where $\varepsilon_j(x)$ are functions (2.31).

Theorem 3.1 *There exists a linear bounded extension operator*

$$E_\sigma^1 : V^1(\Omega, \Gamma_0) \to X_\sigma^1(G) \tag{3.6}$$

(i.e. $E_K(v)(x) \equiv v(x)$ for $x \in \Omega$).

Proof see in [F3].

We show the conditions on $v \in V^1(G)$ which guarantee inclusion $v \in V^1(\Omega,\Gamma_0)$

Theorem 3.2 *Let domains Ω and G satisfy Conditions 3.1, 3.2 and $\partial\Gamma \neq \emptyset$. Suppose that $v(x) \in V^1(\Omega)$ satisfies the conditions: $v|_{\Gamma_0} = 0$ and*

$$v(x) \in \left(H^m(\Omega \cap O(\partial\Gamma))\right)^2 \text{ with } m > \frac{3\alpha - 1}{2}, \ \alpha \geqslant 2, \tag{3.7}$$

where $O(\partial\Gamma)$ is a neighbourhood of $\partial\Gamma$ and α is a magnitude from (3.3). Then $v(x) \in V^1(\Omega, \Gamma_0)$, i.e. it can be extended up to a vector field $Lv \in V_0^1(G)$.

Proof see in [F3].

3 Result on stabilization

We prove now the main theorem of this section on stabilizability by feedback boundary control of 2D Ozeen equations.

Theorem 3.3 *Let domains Ω and G satisfy Conditions 3.1, 3.2. Then for each initial condition $v_0(x) \in V^1(\Omega, \Gamma_0)$ and for each $\sigma > 0$ there exists a feedback control u defined on Σ such that the solution $v(t, x)$ of (2.1)–(2.3), (2.4) satisfies the inequality*

$$\|v(t, \cdot)\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as} \quad t \to \infty. \tag{3.8}$$

Proof. We can assume that σ satisfies condions (2.26), otherwise making it a little bit more. We act to initial condition $v_0 \in V^1(\Omega, \Gamma_0)$ by the operator E_σ^1 from (3.6) and by Theorem 3.1 we obtain that $w_0 = E_\sigma^1 v_0 \in X_\sigma^1(G)$. Since $X_\sigma^1(G) \subset V_0^1(G) \subset V_0^0(G)$, the solution $w(t, x)$ of problem (2.1)–(2.4) can be written in the form $w(t, \cdot) = e^{-At} w_0$ where A is operator (2.15). Hence, (2.28) implies (3.8) $\square$

3 Feedback property.

We show that the method of stabization proposed above can react on unpredictable fluctuations of a system. The point is that if the solution $(v(t,x), u(t,x))$ of problem (2.1)–(2.4) satisfies equality (2.12) and if at time moment $\tilde{t}_0$ the system (2.1)–(2.3) is subjected by certain fluctuation, then $v(t,x)$ at $t = \tilde{t}_0$ is pushed out $\gamma_\Omega X^1_\sigma(G)$ and that is why it will not tend to zero with prescribed rate but, generally saying, it will increase. Therefore we must return it to $\gamma_\Omega X^1_\sigma(G)$ with help of feedback control.

Thus we suppose that the velocity $v(t,x) = \gamma_\Omega w(t,x)$ obtained in Theorem 3.3 is observable, and it is subjected at instants $0 < t_1 < t_2 < \ldots < t_k < \ldots$ by some fluctuations

$$(t_1 - 0)v_{01}(x), (t_2 - t_1)v_{02}(x), \ldots, (t_k - t_{k-1})v_{0k}(x), \ldots \tag{3.9}$$

so as we have, for instance, in a neighbourhood of $t = t_1$:

$$v(t_1 - 0, x) = \gamma_\Omega w(t_1, x), \qquad v(t_1, x) = \gamma_\Omega w(t_1, x) + t_1 v_{01}(x).$$

Note that we do not know in advance neither instants t_j nor fluctuations $v_{oj}(x)$ [2] but we assume that

$$v_{0j}(x) \in V^1(\Omega, \Gamma_0) \quad \forall j. \tag{3.10}$$

Mark also that we put factor $(t_j - t_{j-1})$ before $v_{0j}(x)$ in (3.9) for convenience of normalization.

We organize the reaction of our control on these unpredictable fluctuations by the following way. Let $w_j(t,x), \nabla p_j(t,x)$ be a solution of the boundary value problem

$$\partial_t w_j(t,x) - \Delta w_j + (a(x), \nabla)w_j + (w_j, \nabla)a + \nabla p_j(t,x) = 0, \ \operatorname{div} w_j = 0, \tag{3.11}$$

$$w_j(t,x)|_{t=0} = E^1_\sigma v_{0j}(x) \tag{3.12}$$

where $t > t_j, x \in G$, E^1_σ is operator (3.6) of extension. After that we set

$$\tilde{w}(t,x) = w(t,x) + \sum_{j \in J_t} (t_j - t_{j-1}) w_j(t,x), \qquad \text{where } J_t = \{i : t_i < t\} \tag{3.13}$$

where $w(t,x)$ is the solution of problem (2.10)–(2.12), with $w_0 = E^1_\sigma v_0$ and $w_j(t,x)$ are solutions of problem (3.11), (3.12). Now we define the solution of stabilization problem (2.1)–(2.4) subjected by unpredictable fluctuations (3.9) that are suppressed by feedback control with help of the formula:

$$(\tilde{v}(t,x), \tilde{u}(t,x)) = (\gamma_\Omega \tilde{w}(t,x), \gamma_\Gamma \tilde{w}(t,x)). \tag{3.14}$$

We give an estimate of this solution in the following Theorem:

[2] There is no necessity to know the reasons caused these fluctuations. Nevertheless we can suppose that solution of our equations plus fluctuations describe a real fluid flow

Theorem 3.4 *Let instants* $0 < t_1 < \ldots < t_k < \ldots$ *and unpredictable fluctuations* (3.9) *are unknown in advance but satisfy the inequality*

$$\|v_{0j}\|_{V^1(\Omega,\Gamma_0)} \leqslant \varepsilon, \quad \forall j \tag{3.15}$$

with sufficiently small $\varepsilon > 0$. *Then solution* (3.14) *of stabilization problem (2.1)–(2.4) subjected by unpredictable fluctuations* (3.9) *satisfies the estimate:*

$$\|\tilde{v}(t,\cdot)\|_{V^1(\Omega,\Gamma_0)} \leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + (1+1/\sigma)\varepsilon) \tag{3.16}$$

where $\sigma > 0$ *satisfies (2.26), constant* $c > 0$ *depends on* σ *only,* $\|E^1_\sigma\|$ *is the norm of operator (3.6),* $\varepsilon > 0$ *is the magnitude from inequality* (3.15).

Proof. We estimate vector field (3.13) using upperbound (2.28) and (3.12), (3.15). As a result we get:

$$\|\tilde{w}(t,\cdot)\|_{V^1_0(G)} \leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + \sum_{j\in J_t}(t_j - t_{j-1})e^{-\sigma(t-t_j)}\|v_{0j}\|_{V^1(\Omega,\Gamma_0)})$$

$$\leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + \varepsilon\sum_{j\in J_t}[(t_j - t_{j-1})e^{-\sigma(t-t_{j-1})} + \sigma\int_{t_{j-1}}^{t_j} e^{-\sigma(t-\tau)}\,d\tau]$$

$$\leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + (1+\sigma)\varepsilon\int_0^t e^{-\sigma(t-\tau)}\,d\tau)$$

This inequalities imply (3.16). □

Remark 3.1 Evidently, a velocity of real fluid flow differs from corresponding solution of the Ozeen equations which simulates this real fluid flow on some "noise", i.e. on some small vector field. In our case the solution $\tilde{v}(t,x)$ subjected by fluctuations (3.9) plays the role of the velocity for real fluid flow and fluctuations (3.9) plays the role of this "noise". We suppose that Ozeen equations gives good approximation of reality. Therefore assumption (3.15) is reasonable, and inequality (3.16) has the following meaning: The solution $\tilde{v}(t,x)$ of stabilization problem for velocity of real fluid flow consists of a term which tends to zero as $e^{-\sigma t}$ when $t \to \infty$ and of a "noise" possessing amplitude $c\|E^1_\sigma\|(1+1/\sigma)\varepsilon$.

Remark 3.2 The factor $c\|E^1_\sigma\|(1+1/\sigma)$ in the "noise's" amplitude $c\|E^1_\sigma\|(1+1/\sigma)\varepsilon$ can be very large for large $\sigma > 0$. We can minimize this factor by the following way. We construct a feedback control which suppresses unpredictable fluctuations with help of boundary value problem (3.11), (3.12). In virtue of linearity of Ozeen equations this construction does not depend on construction of stabilization problem given in Theorem 3.3. That is why we can replace operator E^1_σ in (3.12) on analogous operator $E^1_{\sigma_0}$ with $\sigma_0 > 0$ that minimises the function $f(\sigma) = c\|E^1_\sigma\|(1+1/\sigma)$: $f(\sigma_0) = \min\limits_{0<\eta<\sigma} f(\eta)$. (If this σ_0 does not satisfy condition (2.26) we change it on some close positive magnitude which satisfies (2.26).)

Simple generalization allows to pass from "noise" (3.9) which is discrete with respect to time to the "noise" continuous in time. Indeed, let us formally pass to limit in (3.9) as $\Delta \equiv \max_j (t_j - t_{j-1})$ tends to zero. As a result we get differential form

$$v_{0\tau}(x)\, d\tau \tag{3.17}$$

with respect to time τ which coefficient is a vector field satisfying the following analog of (3.10), (3.15):

$$v_{0\tau}(x) \in V^1(\Omega, \Gamma_0) \text{ a.a. } \tau \geqslant 0; \quad \sup_{k\geqslant 0} \int_k^{k+1} \|v_{0\tau}\|_{V^1(\Omega,\Gamma_0)}\, d\tau \leqslant \varepsilon \tag{3.18}$$

(a.a. means "for almost all"). Analogously to (3.11), (3.14) we construct the feedback control that suppressed unpredictable fluctuations (3.17) by the formulas:

$$\partial_t w_\tau(t,x) - \Delta w_\tau + (a(x), \nabla) w_\tau + (w_\tau, \nabla) a + \nabla p_\tau(t,x) = 0, \ \operatorname{div} w_\tau = 0, \tag{3.19}$$

where $t > \tau, x \in G$,

$$w_\tau(t,x)|_{t=0} = E^1_\sigma v_{0\tau}(x) \tag{3.20}$$

$$\tilde{w}(t,x) = w(t,x) + \int_0^t w_\tau(t,x)\, d\tau \tag{3.21}$$

and desired solution $(\tilde{v}(t,x), \tilde{u}(t,x'))$ of stabilization problem (2.1)–(2.4) subjected with fluctuations (3.17) is defined by (3.14).

Theorem 3.5 *Let $(\tilde{v}, \tilde{u})$ be solution* (3.14) *of stabilization problem* (2.1)–(2.4) *subjected by unpredictable fluctuations* (3.17) *and this fluctuations satisfy* (3.18) *with sufficiently small $\varepsilon > 0$. Then $\tilde{v}(t,x)$ satisfies the estimate:*

$$\|\tilde{v}(t,\cdot)\|_{V^1(\Omega,\Gamma_0)} \leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + (1 + 1/(1-e^{-\sigma}))\varepsilon) \tag{3.22}$$

Proof. As well as in the proof of Theorem 3.4 we get from (3.19)–(3.21), (2.28):

$$\|\tilde{w}(t,\cdot)\|_{V^1_0(G)} \leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + \int_0^t e^{-\sigma(t-\tau)}\|v_{0\tau}\|_{V^1(\Omega,\Gamma_0)}\, d\tau)$$

$$\leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + \sum_{k=1}^{[t]} e^{-\sigma(t-k)} \int_{k-1}^k \|v_{0\tau}\|_{V^1(\Omega,\Gamma_0)}\, d\tau + \int_{[t]}^t \|v_{0\tau}\|_{V^1(\Omega,\Gamma_0)}\, d\tau)$$

$$\leqslant c\|E^1_\sigma\|(e^{-\sigma t}\|v_0\|_{V^1(\Omega,\Gamma_0)} + (1 + \sum_{k=1}^{[t]} e^{-\sigma(t-k)})\varepsilon)$$

Inequality (3.22) follows from this estimates. □

4 Stabilization of 2D Navier–Stokes equations

4 Formulation of the stabilization problem

We consider the Navier–Stokes equations

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)v + \nabla p(t,x) = f(x), \quad (t,x) \in Q \tag{4.1}$$

$$\operatorname{div} v = 0 \tag{4.2}$$

($v = (v_1, v_2)$) with initial and boundary conditions

$$v(t,x)|_{t=0} = v_0(x), \quad x \in \Omega \tag{4.3}$$

$$v|_{\Sigma_0} = 0, \quad v|_{\Sigma} = u, \tag{4.4}$$

where $u = (u_1, u_2)$ is a control defined on Σ.

We suppose also that a steady-state solution $(\hat{v}(x), \nabla\hat{p}(x))$ of Navier—Stokes system with the same right-hand side $f(x)$ as in (4.1) is given:

$$\Delta\hat{v}(x) + (\hat{v},\nabla)\hat{v} + \nabla\hat{p} = f(x), \quad \operatorname{div}\hat{v}(x) = 0, \quad x \in \Omega \tag{4.5}$$

$$\hat{v}|_{\Gamma_0} = 0. \tag{4.6}$$

Let $\sigma > 0$ be given. The problem of stabilization with the rate σ is to look for a solution $(v, \nabla p, u)$ of problem (4.1)–(4.4) which satisfies the inequality

$$\|v(t,\cdot) - \hat{v}(\cdot)\|_{(H^1(\Omega))^2} \leqslant ce^{-\sigma t} \quad \text{as } t \to \infty. \tag{4.7}$$

The important additional condition is that u is a feedback control. Definition of feedback notion is analogous to Definition 3.1. Nevertheless we give now exact formulation of feedback property.

We extend Ω to a domain G throught Γ such that Condition 4.2 holds. After that we extend problem (4.1)–(4.4) to analogous problem defined on $\Theta = \mathbb{R}_+ \times G$:

$$\partial_t w(t,x) - \Delta w + (w,\nabla)w + \nabla q(t,x) = g(x), \quad \operatorname{div} w(t,x) = 0 \tag{4.8}$$

$$w(t,x)|_{t=0} = w_0(x) \tag{4.9}$$

with additional condition

$$w|_S = 0 \tag{4.10}$$

Moreover we assume that solution $(\hat{v}, \nabla\hat{p})$ of (4.5), (4.6) is extended on G up to a pair $(a(x), \nabla g(x))$, $x \in G$ such that

$$-\Delta a(x) + (a,\nabla)a + \nabla\hat{q}(x) = g(x), \quad \operatorname{div} a(x) = 0, \quad x \in G \tag{4.11}$$

$$a|_{\partial G} = 0 \tag{4.12}$$

where right side $g(x)$ is the same as in (4.8).

Definition 4.1 A control $u(t,x)$ in stabilization problem (4.1)–(4.4) is called feedback if the solution $(v(t,x), u(t,x))$ of (4.1)–(4.4) is defined by the equality:

$$(v(t,x), u(t,x)) = (\gamma_\Omega w(t,\cdot), \gamma_\Gamma w(t,\cdot)) \tag{4.13}$$

where $w(t,x)$ satisfies to (4.8), (4.10), and γ_Ω, γ_Γ are operators of restriction of a function defined on G to Ω and to Γ respectively.

4 Invariant manifolds

Let $g(x)$ from (4.8) satisfies the condition:

$$g(x) \in (L_2(G))^2. \tag{4.14}$$

Then as well-known (see, for instance [VF]) equations (4.8) are equivalent to the following equation with respect to one unknown function $w(t,x)$:

$$\partial_t w(t,x) - \pi\Delta w + \pi(w,\nabla)w = \pi g(x) \tag{4.15}$$

where π is orthoprojector (2.14) on $V_0^0(G)$ (see (2.13)). We assume in addition that solution w of (4.15) (as well as soulution w of (4.8) belongs to the space

$$V_0^{1,2}(\Theta_T) \equiv \{w(t,x) \in L_2(0,T;V^2(G)\cap(H_0^1(G))^2) : \partial_t w \in L_2(0,T;V_0^0(G)\} \tag{4.16}$$

for each $T > 0$, where $\Theta_T = (0,T) \times G$. It is proved (see [La]) that for each $T > 0$, $g(x) \in (L_2^0(G))^2$, $w_0(x) \in V_0^1(G)$ there exists unique solution $w(t,x) \in V_0^{1,2}(Q_T)$ of problem (4.15), (4.9). Solution $w(t,x)$ of (4.15), (4.9) taken at time moment t we denote as $S(t,w_0)(x)$:

$$w(t,x) = S(t,w_0)(x). \tag{4.17}$$

Since embedding $V^{1,2}(Q_T) \subset C(0,T;V_0^1(G))$ is continuous, the family of operators $S(t,w_0)$ is continuous semigroup on the space $V_0^1(G) : S(t+\tau,w_0) = S(t,S(\tau,w_0))$.

Note that we can rewrite (4.11) in the form analogous to (4.15):

$$-\pi\Delta a(x) + \pi(a,\nabla)a = \pi g, \quad a(x) \in V^2(G) \cap V_0^1(G). \tag{4.18}$$

Since $a(x)$ is steady-state solution of (4.15), $S(t,a) = a$ for each $t \geqslant 0$. We can decompose semigroup $S(t,w_0)$ in a neighbourhood of a in the form

$$S(t,w_0 + a) = a + L_t w_0 + B(t,w_0) \tag{4.19}$$

where $L_t w_0 = S'_w(t,a)w_0$ is derivative of $S(t,w_0)$ with respect to w_0 at point a, and $B(t,w_0)$ is nonlinear operator with respect to w_0. Differentiability of $S(t,w_0)$ is proved, for instance in [BV, Ch. 7. Sect. 5]. Therefore

$$B(t,0) = 0, \quad B'_w(t,0) = 0. \tag{4.20}$$

Moreover in [BV, Ch. 7. Sect. 5] is proved that $B(t,w)$ belongs to class C^α with $\alpha = 1/2$ with respect to w. This means that for each $w_0 \in V_0^1(G)$

$$\|B'_w(t,w_0)\|_{C^\alpha} \equiv \sup_{0\leqslant\|u-w_0\|_{V_0^1(G)}\leqslant 1} \frac{\|B'_w(t,u) - B'_w(t,w_0)\|_{V_0^1(G)}}{\|u-w_0\|^\alpha_{V_0^1(G)}} < \infty$$

and left side is a continuous function with respect to w.

We study now semigroup $L_t w_0 = S'_w(t,a)w_0$ of linear operators. First of all note that $w(t,x) = L_t w_0$ is the soulution of problem (2.8)–(2.10) in which the coefficient a is the soulution of (4.18). Therefore

$$L_t w_0 = e^{-At}w_0 \tag{4.21}$$

where A is Ozeen operator (2.15).

Below we suppose that $r_0 \in (0,1)$ satisfies the property:

$$\{\zeta \in \mathbb{C} : |\zeta| = r_0\} \cap \Sigma(e^{-At_0}) = \emptyset \tag{4.22}$$

where, recall, $\Sigma(e^{-At})$ is the spectrum of operator (4.21).

It is clear, that $\zeta_j \in \Sigma(e^{-At_0})$ if and only if $\zeta_j = e^{-\lambda_j t_0}$ and $-\lambda_j \in \Sigma(-A)$. That is why condition (4.22) is equivalent to condition (2.35) where $\sigma = -\ln r_0/t_0$. Besides, if $|\zeta_j| > r_0$ then $-\mathrm{Re}\lambda_j > -\sigma$.

The following assertion holds:

Theorem 4.1 *Family of operators $e^{-At} : V_0^1(G) \to V_0^1(G)$ where A is operator* (2.15) *is well defined for each $t \geqslant 0$. Let*

$$\sigma_+ = \{\zeta_1, \ldots, \zeta_N : \zeta_j \in \Sigma(e^{-At_0}), \;\; |\zeta_j| > r_0, \; j = 1, \ldots, N\} \tag{4.23}$$

where $r_0 \in (0,1)$ and satisfies (4.22). *Let $X_+ \subset V_0^1(G)$ be the invariant subspace for e^{-At_0} corresponding to σ_+, $\Pi_+ : V_0^1(G) \to X_+$ be the projector on X_+ (i.e., $\Pi_+ V_0^1(G) = X_+$, $\Pi_+^2 = \Pi$) and $X_- = (I - \Pi_+)V_0^1(G)$ be complementary invariant subspace. Let $L_{t_0}^+ = e^{-At_0}|_{X_+} : X_+ \to X_+$, $L_{t_0}^- = e^{-At_0}|_{X_-} : X_- \to X_-$. Then operator $L_{t_0}^+$ has inverse operator $(L_{t_0}^+)^{-1}$. For some t_0 there exist constants $\hat{r}$, ε_+, $\varepsilon_- \in (0,1)$ such that*

$$\|L_{t_0}^-\| \leqslant \hat{r}(1 - \varepsilon_-), \quad \|(L_{t_0}^+)^{-1}\| \leqslant \hat{r}^{-1}(1 - \varepsilon_+). \tag{4.24}$$

Generally speaking eigenvalues of operators A and e^{-At} are complex-valued. That is why all spaces in Theorem 5.1 are complex. But to apply obtained results to (nonlinear) Navier—Stokes equation we need to have analogous results for the real spaces of the same type. Actually, for this we have to define the projector of Π_+ in real spaces.

Lemma 4.1 *Restriction of operator Π_+ on the real space $V_0^1(G)$ can be written in the form*

$$(\Pi_+)(x) = \sum_{j=1}^{K} e_j(x) \int_G v(x)\varepsilon_j(x)\,dx \tag{4.25}$$

where $\{\varepsilon_j\}$ is the set of functions (2.31) which are suitably renumbered and renormalized functions (2.30) and $\{e_j\}$ is set of Real and Imaginary parts of functions (2.22) analogously renumbered and renormalized.

The proof of this simple lemma one can find in [F2].

Lemma 4.2 *For an arbitrary subdomain $\omega \subset G$ vector fields $\{e_j(x),\, j = 1, \ldots, K\}$ from* (4.25) *restricted on ω are linear independent over $\mathbb{R}$.*

Using (4.25) we can easily restrict spaces X_+ and X_- as well as operators $L_{t_0}^+$, $L_{t_0}^-$ defined in formulation of Theorem 4.1 on the real subspaces of $V_0^1(G)$. We denote this new real spaces and operators also by X_+, X_-, $L_{t_0}^+$, $L_{t_0}^-$. This will not lead to misunderstanding because below we do not use their complex analogs.

In a neighbourhood of steady-state solution a of (4.18) we establish existence of a manifold M_- which is invariant with respect to semigroup $S(t,w)$ (i.e., $S(t,w) \in M_-$ $\forall w \in M_- \forall t > 0$). This manifold can be represented as the graph:

$$M_- = \{u \in V_0^1(G) \,:\, u = a + u_- + g(u_-), \;\; u \in X_- \cap \mathcal{O}\} \qquad (4.26)$$

where $\mathcal{O}$ is a neighbourhood of origin in $V_0^1(G)$, $g : X_- \cap \mathcal{O} \to X_+$ is an operator-function of class $C^{3/2}$ and

$$g(0) = 0, \;\; g'(0) = 0. \qquad (4.27)$$

Note that condition (4.27) means that manifold (4.26) is tangent to X_- at point a.

The following theorem is true.

Theorem 4.2 *Let a satisfy* (4.18), *$\sigma > 0$ satisfy* (2.26), *$\mathcal{O} = \mathcal{O}_\varepsilon = \{v \in V_0^1(G) : \|v\|_{V_0^1(G)} < \varepsilon\}$ and ε is sufficiently small. Then there exists unique operator-function $g : X_- \cap \mathcal{O} \to X_+$ of class $C^{3/2}$ satisfying* (4.27) *such that the manifold M_- defined in* (4.26) *is invariant with respect to semigroup $S(t, w_0)$ connected with* (4.15)[3]. *There exists a constant $c > 0$ such that*

$$\|S(t, w_0) - a\|_{V_0^1(G)} \leqslant c\|w_0 - a\|_{V_0^1(G)} e^{-\sigma t} \;\; \text{as } t \geqslant 0 \qquad (4.28)$$

for each $w_0 \in M_-$.

This theorem follows form results of [BV, Ch. 5, Sect. 2; Ch. 7, Sect. 5] and from Theorem 4.1.

4 Extension operator.

Here we construct extension operator for Navier—Stokes equations. This operator is nonlinear analog of extension operator (3.6) constructed for Ozeen equatios.

Recall that the domain Ω and its extension G satisfy Conditions 3.1, 3.2. Besides, the space $V^1(\Omega, \Gamma_0)$ is defined in (3.4).

Theorem 4.3 *Suppose that $a(x)$ is a steady-state solution of* (4.18), *$\hat{v}(x) = \gamma_\Omega a$, and M_- is the invariant manifold constructed in a neighbourhood $a + \mathcal{O}$ of a in $V_0^1(G)$ in Theorem 4.2. Let $B_{\varepsilon_1} = \{v_0 \in V^1(\Omega, \Gamma_0) : \|v_0 - \hat{v}\|_{V^1(\Omega,\Gamma_0)} < \varepsilon_1\}$. Then for sufficiently small ε_1 one can constrct a continuous operator*

$$\text{Ext} : \hat{v} + B_{\varepsilon_1} \to M_-, \qquad (4.29)$$

which is operator of extension for vector fields from Ω to G:

$$(\text{Ext}v)(x) \equiv v(x), \quad x \in \Omega. \qquad (4.30)$$

The proof of this theorem see in [F3].

[3] I.e. $S(t, w_0)$ is the resolving semigroup of equation (4.15) (see (4.17).

4 Theorem on stabilization

We set

$$V^2(\Omega,\Gamma_0) = \{v(x) \in V^2(\Omega) \, : \, v|_{\Gamma_0} = 0, \ \exists w \in V^2(G) \cap V_0^1(G), \ v(x) = \gamma_\Omega w\} \quad (4.31)$$

where γ_Ω is the operator of restriction on Ω.

Proposition 4.1 *Let $f \in (L_2(\Omega))^2$ and a pair $(\hat{v}(x), \nabla\hat{p}(x))$ belongs to $V^2(\Omega,\Gamma) \times (L_2(\Omega))^2$ and satisfies equations* (4.5), (4.6). *Then there exist an extension $g(x) \in (L_2(G))^2$ of $f(x)$ from Ω to G and an extension $(a(x), \nabla q(x))in(V^2(G) \cap V_0^1(G)) \times (L_2(G))^2$ of $(\hat{v}(x), \nabla\hat{p}(x))$ ¿from Ω to G such that the pair $(a(x), \nabla q(x))$ is a solution of* (4.11), (4.12).

We now are in position to formulate the main result of this paper.

Theorem 4.4 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^∞-boundary $\partial\Omega$ and $\partial\Omega = \Gamma_0 \cup \Gamma \cup \partial\Gamma$, where Γ, Γ_0 are open curves, $\Gamma \neq \emptyset$, $\partial\Gamma$ is a finite number of points or $\partial\Gamma = \emptyset$. Suppose that a domain $G \subset \mathbb{R}^2$ is chosen such that Conditions* 3.1, 3.2 *with $\alpha \geqslant 3$ are fulfilled. Let $f(x) \in (L_2(\Omega))^2$ and $(\nabla\hat{v}(x), \nabla\hat{p}(x)) \in V^2(\Omega,\Gamma_0) \times (L_2(\Omega))^2$ satisfy* (4.5), (4.6). *Then for an arbitrary $\sigma > 0$ there exists sufficiently small $\varepsilon > 0$ such that for each $v_0 \in V^1(\Omega,\Gamma_0)$ satisfying*

$$\|\hat{v} - v_0\|_{V_0^1(\Omega)} < \varepsilon_1 \quad (4.32)$$

there exists a feedback boundary control $u(t,x)$, $(t,x) \in \Sigma \equiv \mathbb{R}_+ \times \Gamma$ which stabilized Navier—Stokes boundary value problem (4.1)–(4.4) *with the rate* (4.7), *i.e. the soulution v of* (4.1)–(4.4) *satisfies* (4.7).

The proof of this theorem is based on Theorems 4.2, 4.3 (see [F2],[F3]).

The construction of feedback control suppressing unpredictable fluctuations in the case of nonlinear Navier-Stokes equations will be exposed in some other paper.

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Approximation of Weak Limits via Method of Averaging with Applications to Navier-Stokes Equations

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Abstract

We prove, firstly, that weak limit of some sequence from Orlicz function space can be approximated in strong sense (in norm) by the subsequence of averaged functions if the radius of averaging tends to zero slowly enough. Secondly, we consider the weakly converging sequence of approximate solutions to the Navier-Stokes equations and obtain strong convergence of some subsequence of solutions to averaged equations to the solution of the limiting equations.

Keywords: Orlicz function space, weak limit, averaging operator, strong convergence, embedding theorems.

1 Notations and basic notions from Orlicz function spaces theory.

Let $\Omega \subset \mathbb{R}^d$ be bounded domain with smooth boundary Γ, and $x = (x_1, \dots, x_d)$ be the points of Ω. By $L^1(\Omega)$ we denote the space of absolutely integrable functions on Ω, $L^\infty(\Omega)$– the space of essentially bounded functions and $L^p(\Omega)$, $1 < p < \infty$ – the scale of Lebesgue spaces of functions which are integrable in power p.

We shall use also the Orlicz function spaces, and remind the basic notions (see[1]). Let m(r) be defined on $[0, \infty)$ function, continuous from the right, non-negative, non-decreasing and such that

$$m(0) = 0, \quad m(r) \to \infty \text{ as } r \to \infty. \tag{1}$$

The convex function (Young function)

$$M(t) = \int_0^t m(r)dr \tag{2}$$

produces Orlicz class $K_M(\Omega)$ containing the functions $f(x) \in L^1(\Omega)$ such that $M(|f(x)|)$ belong to $L^1(\Omega)$,too. The linear span of $K_M(\Omega)$ endowed with the norm

$$||f||_{L_M(\Omega)} = \inf \left\{ \lambda > 0 \,\middle|\, \int_\Omega M\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\} \tag{3}$$

is called as Orlicz space $L_M(\Omega)$ associated with Young function $M(t)$. The closure of $L^\infty(\Omega)$ in the norm (3) yields, in general case, another Orlicz space $E_M(\Omega)$, and the inclusions are valid

$$E_M(\Omega) \subseteq K_M(\Omega) \subseteq L_M(\Omega). \tag{4}$$

One says function $M(t)$ satisfies Δ_2 – *condition* (cf.[1]) if there exist constants $C > 0$ and $t_0 > 0$ such that

$$M(2t) \leq C\ M(t) \quad \text{for } \forall t \geq t_0. \tag{5}$$

Three sets E_M, K_M and L_M coincide if and only if $M(t)$ satisfies Δ_2-condition.

In many cases it's possible to compare two Young functions $M_1(t)$ and $M_2(t)$ with respect to their behaviour at infinity , namely, $M_2(t)$ dominates $M_1(t)$ if there exist constants $C > 0$ and $\alpha > 0$ such that

$$M_1(\alpha t) \leq M_2(t) \quad \text{for } \forall t \geq t_0 = t_0(\alpha, C). \tag{6}$$

In this case

$$E_{M_2} \subseteq E_{M_1} \text{ and } L_{M_2} \subseteq L_{M_1}.$$

In particular, function $M(t)$ satisfying Δ_2-condition (5) is dominated by some power function $t^q, q > 1$.

If each function M_k, $k = 1, 2$ dominates the other one, then M_1 and M_2 are equivalent, $M_1 \cong M_2$, and corresponding Orlicz spaces are the same, $E_{M_1} = E_{M_2}$ and $L_{M_1} = L_{M_2}$.

Function M_2 dominates M_1 essentially if

$$\lim_{t \to \infty} \frac{M_1(\beta t)}{M_2(t)} = 0, \quad \forall \beta = \text{const} > 0. \tag{7}$$

In this case the strong embeddings

$$E_{M_2} \subset E_{M_1}, \quad L_{M_2} \subset L_{M_1} \tag{8}$$

take place.

Denote by

$$n(r) = m^{-1}(r) \tag{9}$$

the inverse function to $m(r)$, i.e. $n(m(r)) \equiv r, \forall r > 0$, and introduce Young function

$$N(t) = \int_0^t n(r)dr. \tag{10}$$

This function is called as complementary (adjoint) convex function to $M(t)$, and it's equivalent to the next one:

$$N(t) \cong \sup_{r>0} \{tr - M(r)\}. \tag{11}$$

Two Orlicz spaces L_M and L_N are supplementary, and for any $f(x) \in L_M(\Omega)$, $g(x) \in L_N(\Omega)$ there exists the integral

$$< f, g > \equiv \int_\Omega f(x)g(x)dx \tag{12}$$

which defines the linear continuous functional on $E_N(\Omega)$ with fixed $f \in L_M(\Omega)$ and any $g \in E_M(\Omega)$. It gives the notion of weak convergence in $L_M(\Omega)$: the sequence $\{f_n(x)\}$ converges weakly to $f(x) \in L_M(\Omega)$,
$f_n \rightharpoonup f$, if

$$< f_n, g > \to < f, g > \quad \text{for each } g \in E_N(\Omega) \quad as \quad n \to \infty. \tag{13}$$

At the same time it's possible to define mean convergence in $L_M(\Omega)$:

$$f_n \to f \quad \text{in mean value, if}$$

$$\int_\Omega M(|f_n - f|)dx \to 0 \quad \text{as } n \to \infty.$$

If $M(t)$ satisfies Δ_2-condition then the mean convergence is equivalent to the strong convergence, i.e. in the norm of $L_M(\Omega)$. Otherwise, mean convergence is stronger than weak, but weaker than strong one.

As an important and interesting examples we shall keep in mind three Young functions: $M_1(t) = t^p$, $1 < p < \infty$, $M_2(t) = e^t - t - 1$ and $M_3(t) = (1+t)\ln(1+t) - t$. The first function $M_1(t)$ yields the Lebesgue space $L^p(\Omega)$, the second one $M_2(t)$ produces two Orlicz spaces L_{M_2} and E_{M_2} because $M_2(t)$ doesn't satisfy Δ_2-condition, and $M_3(t)$ is slowly increasing function which is essentially dominated by$M_1(t)$. The Orlicz space $L_{M_3}(\Omega)$ is located between $L^1(\Omega)$ and any $L^p(\Omega)$, $p > 1$ while $L_{M_2}(\Omega)$ is between $L^p(\Omega), \forall 1 < p < \infty$, and $L^\infty(\Omega)$.

Finally, for any $f(x) \in L^1(\Omega)$ and $h > 0$, let us denote

$$f_h(x) = \frac{1}{h^d} \int\limits_{\Omega} f(y)\omega\left(\frac{x-y}{h}\right) dy \tag{15}$$

an averaging of $f(x)$ where $\omega(z)$ is the kernel of averaging:

$$\omega(z) \in C_0^\infty(\mathbb{R}^d), \quad \omega(z) \geq 0, \quad \int\limits_{\mathbb{R}^d} \omega(z)dz = 1.$$

It's the well-known fact that

$$||f_h - f||_{L_M(\Omega)} \to 0 \quad \text{as } h \to 0 \tag{16}$$

if $M(t)$ satisfies Δ_2-condition, and

$$\int\limits_{\Omega} M(|f_h - f|)dx \to \quad \text{as } h \to 0 \tag{17}$$

for any $M(t)$, i.e.the sequence of $\{f_n\}$ approximates f in the sense of mean convergence.

2 Strong approximation of weak limits

Let us consider some sequence of functions $\{f_n(x)\}$, $n = 1, 2, \ldots$, from Orlicz space $L_M(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L_M(\Omega)$, as $n \to \infty$. For each $f_n(x)$ we construct the family of averaged functions $(f_n)_h(x)$.

Theorem 1 *.If $M(t)$ satisfies Δ_2-condition then there exists subsequence $(f_m)_{h_m}$ such that*

$$(f_m)_{h_m} \to f \quad \textit{strongly in } L_M(\Omega)$$
$$\textit{as } m \to \infty, \ h_m \to 0.$$

For any $M(t)$ there exists subsequence $(f_m)_{h_m}$ such that

$$\int\limits_{\Omega} M(|f - (f_m)_{h_m}|)dx \to 0 \quad \textit{as } m \to \infty, \ h_m \to 0,$$

i.e.mean convergence takes place.

Proof

Step 1. Simple example. In order to understand the problem we consider,firstly, one very simple example of the sequence $f_n(x) = \sin(nx)$, $n = 1, 2, \ldots$, $x \in \mathbb{R}^1$, $\Omega = (0, \pi)$, of periodic functions.In this case we have

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L^2(0, \pi).$$

Let us take the Steklov averaging

$$(f_n)_h(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f_n(\xi) d\xi. \tag{18}$$

It's easy to calculate

$$(f_n)_h(x) = \frac{\sin nh}{nh} \sin(nx). \tag{19}$$

It gives

a) If $h = h_n \to 0$ as $n \to \infty$, but $nh_n \to \infty$ (for example, $h_n = n^{-\alpha}$, $0 < \alpha < 1$) then

$$(f_n)_{h_n}(x) \rightrightarrows 0 \text{ as } n \to \infty,$$

i.e. $(f_n)_{h_n}(x)$ tends to 0 strongly when $h_n \to 0$ slowly enough.

b) If $nh_n \to$ *const* $< \infty$ (or are bounded), then $(f_n)_{h_n} \rightharpoonup 0$ weakly only.

Step 2. One-dimensional case, Steklov averaging. Now consider the case of $d = 1$, $x \in \mathbb{R}^1$, and $f_n(x) \rightharpoonup f(x), n \to \infty$, weakly in $L_M(\Omega)$, where $M(t)$ satisfies Δ_2-condition.

For the sake of simplicity we assume $f_n(x)$, $f(x)$ to be T-periodic, $T = \text{const} > 0$, and, moreover, without loss of generality, one can admit $f(x) \equiv 0$, i.e.

$$f_n(x) \rightharpoonup 0 \text{ weakly in } L_M(0, T). \tag{20}$$

Let us construct the family of functions

$$(f_n)_h(x) = \frac{1}{2h} \int\limits_{x-h}^{x+h} f_n(\xi)\xi \tag{21}$$

and the sequence

$$F_n(x) = \int\limits_0^x f_n(\xi) d\xi$$

which doesn't depend on h. Formula (21) can be rewritten as follows

$$(f_n)_h(x) = \frac{1}{2h}(F_n(x+h) - F_n(x-h)). \tag{23}$$

The sequence $\{F_n(x)\}$ possesses the estimates

$$\sup_x |F_n(x)| \le C, \quad \|F_n'(x)\|_{L_M(0,T)} \le C \tag{24}$$

with constant C independent on n.

Compactness theorem implies $F_n(x) \to 0$ strongly in $L_M(0,T)$, i.e.

$$\|F_{n(x)}\|_{L_M(0,T)} \le C_n, \quad C_n \to 0 \text{ as } n \to \infty. \tag{25}$$

If we take $F_n(x+h)$ or $F_n(x-h$ (displacements of $F_n(x)$) then

$$\|F_n(x+h)\|_{L_M} \le C_n \cdot C, \quad \|F_n(x-h)\|_{L_M} \le C_n \cdot C \tag{26}$$

with C independent on h.

It gives

$$\|(f_n)_h\|_{L_M} \le C \cdot \frac{C_n}{h}. \tag{27}$$

So, if we take $h = h_n$ such that $C_n \cdot h_n^{-1} \to 0$ as $n \to \infty$, for example, $h_n = C_n^\beta$, $0 < \beta < \text{const} < 1$, then

$$(f_n)_{h_n}(x) \to 0 \quad \text{strongly in } L_M(0,T). \tag{28}$$

It proves the theorem 1 in 1-dimensional case.

Step 3. The general case. Let the sequence $\{f_n(x)\}$, $x \in \Omega \subset \mathbb{R}^d$, be weakly converging in $L_M(\Omega)$ to $f(x) \equiv 0$, where $M(t)$ satisfies Δ_2-condition.

We extend $f_n(x)$ by 0 outside of Ω and consider the family $(f_n)_h(x)$ given by the formula (15)

$$(f_n)_h(x) = \frac{1}{h^d} \int\limits_{\mathbb{R}^d} f_n(y)\omega\left(\frac{x-y}{h}\right) dy \tag{29}$$

with arbitrary kernel of averaging $\omega(z)$.

At the beginning we fix some $h = h_0$, for example $h_0 = 1$, and consider the sequence

$$F_n(x) = \int\limits_{\mathbb{R}^d} f_n(y)\omega(x-y)dy \equiv \int\limits_{\mathbb{R}^d} f_n(x-z)\omega(z)dz$$

which doesn't depend on h.

As in one-dimensional case we conclude by the compactness theorem

$$F_m(x) \to 0 \text{ strongly in } L_M(\Omega),$$

i.e. $\|F_m\|_{L_M(\Omega)} \le C_m$, $C_m \to 0$ as $m \to \infty$. If we take the family

$$F_{mh}(x) \equiv \int\limits_{\mathbb{R}^d} f_m(y)\omega(\frac{x-y}{h})dy$$

then for $h \le h_0 = 1$

$$||F_{mh}||_{L_M(\Omega)} \le C||F_m||_{L_M(\Omega)}$$

with constant C independent on h, i.e.

$$||F_{mh}||_{L_M(\Omega)} \le C \cdot C_m.$$

It means that if we choose $h = h_m$ such that

$$C_m h_m^{-d} \to 0 \text{ as } \quad m \to \infty, \ h_m \to 0, \tag{30}$$

for instance, $h_m^d = C_m^\beta$, $0 < \beta < 1$, then $(f_m)_{h_m} \to 0$ strongly in $L_M(\Omega)$.

The theorem 1 is proved for the case of Young function $M(t)$ satisfying Δ_2-condition. For any $M(t)$ the same proof can be used if the mean convergence is considered instead of strong one or the strong convergence in Orlicz space $E_M(\Omega)$ where the set of smooth functions is dense (in particular, the set of averaged functions).

Remark. The main significance of theorem 1 seems to be useful for justification of the smoothing approach in computation of non-smooth solutions to partial differential equations. Big oscillations occur near singularities, and the appearance of oscillations can be connected with the weak convergence of approximate solutions to the exact one. The procedure of "smoothing of solution" means, in fact, an averaging, so, the theorem 1 indicates that the radius of averaging must be big enough according to condition (30).

3 Applications to Navier-Stokes equations

It this section we illustrate the theorem 1 by one example from the theory of Navier-Stokes equations for viscous incompressible fluid. We consider the sequence $\{\mathbf{u}_n\}$ of solutions to Navier-Stokes equations [2]:

$$\begin{gathered} \frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n \cdot \nabla)\mathbf{u}_n + \nabla p_n = \nu\Delta\mathbf{u}_n + \mathbf{f}_n, \\ \operatorname{div}\mathbf{u}_n = 0, \quad (x,t) \in Q = \Omega \times (0,T), \quad \Omega \subset \mathbb{R}^3, \quad \Gamma = \partial\Omega, \end{gathered} \tag{31}$$

complemented with the initial and boundary data

$$\mathbf{u_n}\Big|_{t=0} = \mathbf{u_n^0}(x), \quad x \in \Omega, \quad \mathbf{u_n}\Big|_{\Gamma} = 0. \tag{32}$$

Let us suppose

$$\mathbf{u_n^0}(x) \rightharpoonup \mathbf{u^0}(x) \text{ weakly in } L^2(\Omega),$$

$$\mathbf{f_n} \rightharpoonup \mathbf{f} \text{ weakly in} L^1(0,T;L^2(\Omega))$$

as $n \to \infty$. In view of well-known energy a priori estimate

$$\sup_{0<t<T} \|\mathbf{u}_n(t)\|^2_{L^2(\Omega)} + \int_0^T \|\nabla \mathbf{u}_n(t)\|^2_{L^2(\Omega)} dt \le C \tag{33}$$

with constant C independent on n, we may admit

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;L^2(\Omega)) \tag{34}$$

with any p, $1 \le p < \infty$.

According to the theorem 1 it's possible to extract subsequence $\{(\mathbf{u}_m)_{h_m}\}$ of averaged functions (with respect to all independent variables or to spatial variables only) such that

$$(\mathbf{u}_m)_{h_m} \to \mathbf{u} \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;L^2(\Omega)),$$

$$(\mathbf{f}_m)_{h_m} \to \mathbf{f} \text{ strongly in} L^1(0,T;L^2(\Omega)).$$

And the question arises here: are $\{(\mathbf{u}_m)_{h_m}\}$ the approximate solutions to the limiting equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f}, \quad \mathrm{div}\mathbf{u} = 0 \tag{35}$$

in some strong sense?

To give the answer to this question we apply the operator of averaging to equations (31) and obtain the system (35) for $(\mathbf{u}_m)_{h_m}$ with new right part

$$\mathbf{F}_m = (\mathbf{f}_m)_{h_m} + [((\mathbf{u}_m)_{h_m} \cdot \nabla)(\mathbf{u_m})_{h_m} - ((\mathbf{u}_m \cdot \nabla)\mathbf{u}_m)_{h_m}] \equiv$$

$$(\mathbf{f}_m)_{h_m} + \varphi_{\mathbf{m}}.$$

Theorem 2 . *There exists subsequence $(\mathbf{u}_m)_{h_m}$ such that the difference φ_m tends to zero in the norm of space $L^p(0,T;L^q(\Omega))$ with exponents (p,q), $p \in [1,2]$, $q \in [1,3/2]$, $1/p + 3/2q \ge 2$.*

Proof is the simple corollary of the theorem 1 and embedding theorems.

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Asymptotic Profiles of Nonstationary Incompressible Navier-Stokes Flows in $\mathbb{R}^n$ and $\mathbb{R}^n_+$

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1 Introduction

We are interested in the large-time asymptotic profiles of weak and strong solutions of the Navier-Stokes system in the whole-space $\mathbb{R}^n$ and in the half-space $\mathbb{R}^n_+$, $n \geq 2$:

$$\begin{aligned} &\partial_t u + u \cdot \nabla u = \Delta u - \nabla p \quad (x \in D^n,\ t > 0) \\ &\nabla \cdot u = 0 \qquad\qquad\qquad\quad (x \in D^n,\ t \geq 0) \\ &u|_{t=0} = a. \end{aligned} \tag{NS}$$

Here, $D^n = \mathbb{R}^n$ or $\mathbb{R}^n_+$; and when $D^n = \mathbb{R}^n_+$, we impose the boundary condition

$$u|_{\partial R^n_+} = 0.$$

$u = (u^1, \cdots, u^n)$ and p denote, respectively, unknown velocity and pressure; a is a given initial velocity; and

$$\partial_t = \partial/\partial t, \qquad \nabla = (\partial_1, \cdots, \partial_n), \quad \partial_j = \partial/\partial x_j \quad (j = 1, \cdots, n),$$

$$\Delta u = \sum_{j=1}^n \partial_j^2 u, \qquad u \cdot \nabla u = \sum_{j=1}^n u^j \partial_j u, \qquad \nabla \cdot u = \sum_{j=1}^n \partial_j u^j.$$

We want to find asymptotic profiles of Navier-Stokes flows under some specific conditions on the initial velocities. In Section 2 we state our main results for flows in $\mathbb{R}^n$. The first result that, as $t \to \infty$, the solutions u admit an asymptotic expansion in terms of the space-time derivatives of Gaussian-like functions, provided the initial velocity a satisfies appropriate decay conditions and moment conditions. This improves a result of Carpio [2], in which is deduced the first-order asymptotics of

two kinds, one in $\mathbb{R}^3$ and the other in $\mathbb{R}^2$. We show that one and the same result holds in all dimensions $n \geq 2$.

In Section 3 we state our result for flows in $\mathbb{R}^n_+$. In this case our asymptotic expansion involves only the normal derivatives of Gaussian-like functions in contrast to the case of flows in $\mathbb{R}^n$; but the essential feature is the same. Namely, the functions describing the profiles are of the form $t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, where K stands for some specific functions which are bounded and L^q-integrable for all $1 < q < \infty$. However, it should be emphasized here that in the case of flows in $\mathbb{R}^n$ the functions K are all in L^1, while this is not always true for flows in the half-space. This suggests that the Stokes semigroup over the half-space would never be bounded in L^1. We then apply our expansion result to the analysis of the modes of energy decay of Navier-Stokes flows in the half-space and give (see Corollary 3.6) a characterization of weak solutions which admit the lower bound of rates of energy decay of the form $\|u(t)\|_2^2 \geq ct^{-\frac{n+2}{2}}$. This extends a result of [12] to flows in the half-space.

In Section 4 we give our proof in the case of flows in $\mathbb{R}^n$, and in Section 5 we mainly show how the above-mentioned characterization (Corollary 3.6) is deduced from our asymptotic expansion (Theorem 3.5). Detailed arguments are given in [3], [4].

2 Results for flows in $\mathbb{R}^n$

Consider the Navier-Stokes system in $\mathbb{R}^n$, $n \geq 2$, in the form of the integral equation:

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla \cdot (u \otimes u)(s)ds. \tag{2.1}$$

Here, $A = -\Delta$ is the Laplacian, $u \otimes u = (u^j u^k)_{j,k=1}^n$, $\{e^{-tA}\}_{t\geq 0}$ is the heat semigroup, and P is the bounded projection [7] onto the spaces of solenoidal vector fields. We assume that the initial velocity a is solenoidal, bounded, smooth and satisfies

$$\int (1 + |y|)|a(y)|dy < \infty. \tag{2.2}$$

Assumption (2.2) implies $a \in L^1$; so the divergence-free condition ensures ([10])

$$\int a(y)dy = 0. \tag{2.3}$$

We know ([7], [10]) that a unique strong solution u exists in L^q, $1 \leq q \leq \infty$, satisfying

$$\|u(t)\|_q \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}, \qquad \|\nabla u(t)\|_q \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}, \tag{2.4}$$

if a is small in L^n and satisfies (2.2). Hereafter, $\|\cdot\|_r$ denotes the L^r-norm and $\nabla = (\partial_1, \cdots, \partial_n)$, $\partial_j = \partial/\partial x_j$. The kernel function $P(x) = (P_{jk}(x))$ of the projection P has the Fourier transform

$$\widehat{P}_{jk}(\xi) \equiv \int e^{-ix\cdot\xi}P_{jk}(x)dx = \delta_{jk} + \frac{i\xi_j i\xi_k}{|\xi|^2} \qquad (i = \sqrt{-1}, \quad x \cdot \xi = \sum_{j=1}^n x_j \xi_j). \tag{2.5}$$

So, the kernel function $F_\ell = (F_{\ell,jk})_{j,k=1}^n$ of the operator $e^{-tA}P\partial_\ell$ has the Fourier transform

$$\widehat{F}_{\ell,jk}(\xi,t) = i\xi_\ell e^{-t|\xi|^2}(\delta_{jk} + i\xi_j i\xi_k/|\xi|^2) \equiv \widehat{F}^1_{\ell,jk}(\xi,t) + \widehat{F}^2_{\ell,jk}(\xi,t).$$

Denoting the heat kernel by

$$E_t(x) = (4\pi t)^{-\frac{n}{2}} \exp(-\tfrac{|x|^2}{4t})$$

and writing $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ for any multi-index $\alpha = (\alpha_1,\cdots,\alpha_n)$ of nonnegative integers, we see that $F^1_{\ell,jk}(x,t) = (\partial_\ell E_t)(x)\delta_{jk}$, and so

$$\|\partial_t^p \partial_x^\beta F^1_{\ell,jk}(\cdot,t)\|_q \leq C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\leq q\leq\infty). \tag{2.6}$$

To estimate $F^2_{\ell,jk}$, we use the relation $|\xi|^{-2} = \int_0^\infty e^{-s|\xi|^2}ds$, to get

$$\widehat{F}^2_{\ell,jk}(\xi,t) = i\xi_\ell i\xi_j i\xi_k \int_0^\infty e^{-(s+t)|\xi|^2}ds,$$

$$\text{so that}\quad \partial_x^\beta F^2_{\ell,jk}(x,t) = \int_0^\infty \partial_x^\beta\partial_\ell\partial_j\partial_k E_{s+t}(x)ds.$$

This implies

$$\|\partial_t^p \partial_x^\beta F^2_{\ell,jk}(\cdot,t)\|_q \leq C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\leq q\leq\infty).$$

Combining this with (2.6) gives

$$\|\partial_t^p \partial_x^\beta F_{\ell,jk}(\cdot,t)\|_q \leq C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad (1\leq q\leq\infty). \tag{2.7}$$

Our results for flows in $\mathbb{R}^n$ are the following. Hereafter we use the summation convention.

Theorem 2.1. (i) *Let a be bounded, smooth, solenoidal and satisfy* (2.2). *Let $u = (u^1,\cdots,u^n)$ be the corresponding strong solution of* (2.1). *For $1\leq q\leq\infty$ and $j=1,\cdots,n$, we have*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\Big\|u^j(t) + (\partial_k E_t)(\cdot)\int y_k a^j(y)dy$$
$$+F_{\ell,jk}(\cdot,t)\int_0^\infty\int (u^\ell u^k)(y,s)dyds\Big\|_q = 0. \tag{2.8}$$

(ii) *Suppose $a=(a^1,\cdots,a^n)$ satisfies the following additional conditions:*

$$\int |y|^m|a(y)|dy<\infty, \qquad |a(y)|\leq C(1+|y|)^{-n-1}, \qquad a^j = \sum_{k=1}^n \partial_k b^{jk},$$
$$|b^{jk}(y)|\leq C(1+|y|)^{-n}, \qquad b^{jk} \textit{ are small in } L^1, \tag{2.9}$$

for some integer m with $1 \leq m \leq n$. Then, for $1 \leq q \leq \infty$ and $j = 1, \cdots, n$, we have

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| u_j(t) - \sum_{1\leq|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_x^\alpha E_t)(\cdot) \int y^\alpha a^j(y)dy$$
$$+ \sum_{|\beta|+2p\leq m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p \partial_x^\beta F_{\ell,jk})(\cdot,t) \int_0^\infty \int s^p y^\beta (u^\ell u^k)(y,s)dyds \Bigg\|_q = 0. \tag{2.10}$$

Theorem 2.2. (iii) *For every $a \in L^2$ which is solenoidal and satisfies* (2.2), *there exists a weak solution u which admits the expansion* (2.8) *with $1 \leq q \leq 2$.* (iv) *Let $n = 3, 4$, and suppose that*

$$\int (1+|y|)^{n-1}|a(y)|dy < \infty, \qquad \int (1+|y|)^n |a(y)|^2 dy < \infty. \tag{2.11}$$

Then there exists a weak solution u satisfying (2.10) *for $1 \leq q \leq 2$ and $1 \leq m \leq n-1$.*

Remarks. (i) (2.4) implies $\|u(s)\|_2^2 \leq C(1+s)^{-\frac{n}{2}-1}$, so the last integral in (2.8) is finite.

(ii) Convergence of the integrals in the second sum of (2.10) is ensured by the estimate:

$$\int |y|^m |u(y,s)|^2 dy \leq C(1+s)^{-\frac{n-m}{2}-1} \qquad (0 \leq m \leq n+1), \tag{2.12}$$

which holds if a satisfies (2.9). Estimate (2.12) is deduced in the following way: First, as shown in [10], assumption (2.2) implies $\|u(s)\|_1 \leq C(1+s)^{-\frac{1}{2}}$. Secondly, we know (see [11]) that (2.9) ensures $|y|^{n+1}|u(y,s)| \leq C$ for all $y \in \mathbb{R}^n$ and $s \geq 0$. Therefore,

$$\int |y|^{n+1}|u(y,s)|^2 dy \leq \sup_y(|y|^{n+1}|u(y,s)|) \int |u(y,s)|dy \leq C(1+s)^{-\frac{1}{2}}.$$

Since $\|u(s)\|_2^2 \leq C(1+s)^{-\frac{n}{2}-1}$ by (2.4), we get (2.12) via Hölder's inequality.

(iii) Theorem 2.1 improves an asymptotic result of Carpio [2] in the following sense. First, the result of [2] ignores the vanishing of the average (2.3), and so contains the trivial term $E_t(x) \int a(y)dy \equiv 0$. Secondly, [2] deals only with the case discussed in assertion (i) of Theorem 2.1, but the results given there are incomplete in the two-dimensional case.

(iv) The proof of Theorem 2.2 is almost the same as that of Theorem 2.1. It differs only in estimating the nonlinear convolution integral of (2.1) in a neighborhood of $s = t$. The restriction $m \leq n-1$ in (iv) is needed since for weak solutions we know only that

$$\int |y|^m |u(y,s)|^2 dy \leq C(1+s)^{-(1+\frac{n}{2})(1-\frac{m}{n})} \qquad (0 \leq m \leq n, \quad n = 3, 4), \tag{2.13}$$

which is weaker than (2.12). Estimate (2.13) is due to [6], [13].

3 Results for flows in $\mathbb{R}^n_+$

Consider first the Stokes system

$$\begin{aligned} \partial_t v &= \Delta v - \nabla p && (x \in \mathbb{R}^n_+,\ t > 0) \\ \nabla \cdot v &= 0 && (x \in \mathbb{R}^n_+,\ t \geq 0) \\ v|_{\partial R^n_+} &= 0, \qquad v|_{t=0} = a. \end{aligned} \tag{S}$$

Using the Helmholtz decomposition ([1]):

$$\boldsymbol{L}^r(\mathbb{R}^n_+) \equiv (L^r(\mathbb{R}^n_+))^n = \boldsymbol{L}^r_\sigma \oplus \boldsymbol{L}^r_\pi, \qquad 1 < r < \infty,$$

with

$$\boldsymbol{L}^r_\sigma = \{u \in \boldsymbol{L}^r(\mathbb{R}^n_+)\ :\ \nabla \cdot u = 0,\quad u^n|_{\partial R^n_+} = 0\},$$

$$\boldsymbol{L}^r_\pi = \{\nabla p \in \boldsymbol{L}^r(\mathbb{R}^n_+)\ :\ p \in L^r_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)\},$$

and the associated projection $P = P_r$ onto $\boldsymbol{L}^r_\sigma$, we write the problem (S) in the form

$$\partial_t v + A_r v = 0 \quad (t > 0), \qquad v(0) = a \in \boldsymbol{L}^r_\sigma, \tag{S$'$}$$

where $A = A_r = -P\Delta$ is the Stokes operator defined on

$$D(A_r) = \boldsymbol{L}^r_\sigma \cap \{u \in \boldsymbol{W}^{2,r}(\mathbb{R}^n_+)\ : u|_{\partial R^n_+} = 0\}.$$

We know ([1]) that $-A_r$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $\boldsymbol{L}^r_\sigma$ so that the function $v(t) = (v', v^n) = e^{-tA}a$, $a \in \boldsymbol{L}^r_\sigma$, gives a unique solution of (S$'$) in $\boldsymbol{L}^r_\sigma$. Ukai [15] gave the following representation of the solution v:

$$v^n(t) = Ue^{-tB}[a^n - S \cdot a']; \qquad v'(t) = e^{-tB}[a' + Sa^n] - Sv^n. \tag{3.1}$$

Hereafter, $B = -\Delta$ denotes the Dirichlet-Laplacian on $\mathbb{R}^n_+$; $\{e^{-tB}\}_{t\geq 0}$ is the bounded analytic semigroup in L^p-spaces generated by $-B$; $S = (S_1, \cdots, S_{n-1})$ are the Riesz transforms on $\mathbb{R}^{n-1}$; and U is the bounded linear operator from $\boldsymbol{L}^r(\mathbb{R}^n_+)$ to itself, $1 < r < \infty$, which is defined via the Fourier transform on $\mathbb{R}^{n-1}$ as

$$\widehat{(Uf)}(\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi'|(x_n - y)} \hat{f}(\xi', y)dy. \tag{3.2}$$

Recall that

$$e^{-tB}f = E_t * f^*|_{R^n_+}, \tag{3.3}$$

for a function f defined on $\mathbb{R}^n_+$, where E_t is the heat kernel on $\mathbb{R}^n$ and f^* is the odd extension of the function f defined on $\mathbb{R}^n_+$:

$$f^*(x', x_n) = \begin{cases} f(x', x_n) & (x_n > 0), \\ -f(x', -x_n) & (x_n < 0). \end{cases} \tag{3.4}$$

Let $\|\cdot\|_q$, $1 \le q \le \infty$, denote the norm of $L^q(\mathbb{R}^n_+)$. The following are the standard L^r-L^q estimates for the Stokes semigroup.

$$\|\nabla^k e^{-tA} a\|_q \le C t^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|a\|_r \tag{3.5}$$

with $k = 0, 1, 2 \ldots$, provided either $1 \le r < q \le \infty$, or $1 < r \le q < \infty$; and

$$\|\nabla e^{-tA} a\|_r \le C t^{-\frac{1}{2}}\|a\|_r \qquad (r = 1, \infty). \tag{3.6}$$

Note that in (3.5) the exponents r and q may take on values 1 and ∞, respectively, although the Stokes semigroup would not be bounded in L^1, nor in L^∞. Estimates (3.5) are proved in [1]; and estimates (3.6) are proved in [5] for $r = 1$ and in [14] for $r = \infty$, respectively.

We further need the following estimates:

Proposition 3.1. *Let $a \in L^q_\sigma$ for some $1 < q < \infty$ and*

$$\int_{R^n_+} (1 + y_n)|a(y)|dy < \infty. \tag{3.7}$$

Then,

$$\begin{aligned} \|e^{-tA} a\|_q &\le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\left(\|a\|_q + \int_{R^n_+} y_n |a(y)|dy\right) \\ \|\nabla^j e^{-tA} a\|_r &\le C t^{-\frac{1+j}{2}-\frac{n}{2}(1-\frac{1}{r})}\int_{R^n_+} y_n|a(y)|dy \qquad (1 < r \le \infty, \quad j = 0, 1). \end{aligned} \tag{3.8}$$

Proof. We use representation (3.1) for $e^{-tA}a$. It is easy to see that

$$e^{-tB} S \cdot a' = e^{t\Delta} S \cdot (a')^*,$$

where $e^{t\Delta}$ means convolution with the heat kernel on $\mathbb{R}^n$. The Fourier image of the kernel function of the convolution operator $e^{t\Delta}S$ is $e^{-t|\xi|^2} i\xi'/|\xi'|$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$. Inserting $|\xi'|^{-1} = \pi^{-1/2}\int_0^\infty \eta^{-\frac{1}{2}} e^{-\eta|\xi'|^2} d\eta$ gives

$$e^{-t|\xi|^2}\frac{i\xi'}{|\xi'|} = \pi^{-\frac{1}{2}} i\xi' e^{-t|\xi|^2}\int_0^\infty \eta^{-\frac{1}{2}} e^{-\eta|\xi'|^2} d\eta = \pi^{-\frac{1}{2}} e^{-t\xi_n^2}\int_0^\infty \eta^{-\frac{1}{2}} i\xi' e^{-(\eta+t)|\xi'|^2} d\eta.$$

Thus, the kernel function $F_t = (F_t^1, \cdots, F_t^{n-1})$ of $e^{t\Delta}S$ is

$$F_t(x) \equiv (e^{t\Delta}S)(x) = \pi^{-\frac{1}{2}} E_t(x_n)\int_0^\infty \eta^{-\frac{1}{2}}\nabla' E_{\eta+t}(x')d\eta, \tag{3.9}$$

where $\nabla' = (\partial_1, \cdots, \partial_{n-1})$. It is easy to see that

$$\|\partial_t^\ell \nabla^m F_t\|_p \le C t^{-\frac{m+2\ell}{2}-\frac{n}{2}(1-\frac{1}{p})} \qquad \text{for } 1 < p \le \infty \text{ and } \ell,\ m = 0, 1, \ldots. \tag{3.10}$$

We now prove (3.8). In what follows integration with respect to the space variables will be performed on the whole space $\mathbb{R}^n$ unless otherwise specified. Suppose (3.7) holds. Since $\int_{-\infty}^{\infty} f^*(y', y_n)dy_n = 0$ for a.e. $y' \in \mathbb{R}^{n-1}$ whenever $f \in L^1(\mathbb{R}^n_+)$, direct calculation gives

$$\begin{aligned} e^{t\Delta}(a^n)^* &= \int E_t(x'-y')[E_t(x_n-y_n)-E_t(x_n)](a^n)^*(y)dy \\ &= -\int_0^1\int y_n E_t(x'-y')(\partial_n E_t)(x_n-y_n\theta)(a^n)^*(y)dyd\theta. \end{aligned} \tag{3.11}$$

Application of Minkowski's inequality for integrals yields

$$\|e^{-tB}a^n\|_q \le C\|E_t\|_q\|\partial_n E_t\|_q \int |y_n|\cdot|(a^n)^*(y)|dy \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{R^n_+} y_n|a(y)|dy.$$

Similarly, we get

$$\|e^{-tB}S\cdot a'\|_q \quad \le \quad Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{R^n_+} y_n|a(y)|dy.$$

Since U and S are bounded in $L^r(\mathbb{R}^n_+)$ for $1<r<\infty$, these calculations imply that

$$\|e^{-tA}a\|_q \le Ct^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\int_{R^n_+} y_n|a(y)|dy \qquad \text{for all } 1<q<\infty.$$

On the other hand, we have $\|e^{-tA}a\|_q \le C\|a\|_q$ (3.5); so we obtain

$$\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\left(\|a\|_q + \int_{R^n_+} y_n|a(y)|dy\right).$$

The above argument and (3.5) together yield

$$\|\nabla e^{-tA}a\|_r \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_r \le$$

$$Ct^{-1-\frac{n}{2}(1-\frac{1}{r})}\int_{R^n_+} y_n|a(y)|dy \qquad \text{for all } 1<r<\infty.$$

This proves (3.8) in case $1<r<\infty$. When $r=\infty$, we apply (3.5)–(3.6) to get

$$\begin{aligned} &\|e^{-tA}a\|_\infty \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_n \le Ct^{-\frac{1+n}{2}}\int_{R^n_+} y_n|a(y)|dy, \\ &\|\nabla e^{-tA}a\|_\infty \le Ct^{-\frac{1}{2}}\|e^{-tA/2}a\|_\infty \le Ct^{-1-\frac{n}{2}}\int_{R^n_+} y_n|a(y)|dy. \end{aligned}$$

This proves Proposition 3.1.

We write problem (NS) as

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds \tag{IE}$$

and discuss the existence of weak and strong solutions with specific decay properties that will be treated in our main result.

We first deal with the weak solutions, which are known (see [1], [9]) to exist globally in time for all $a \in L^2_\sigma$, satisfying the identity:

$$\langle u(t),\varphi\rangle = \langle e^{-tA}a,\varphi\rangle + \int_0^t \langle u\otimes u, \nabla e^{-(t-s)A}\varphi\rangle ds \tag{3.12}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n_+)$ with $\nabla\cdot\varphi = 0$ and the energy inequality:

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \le \|a\|_2^2 \qquad \text{for all } t \ge 0. \tag{E}$$

Theorem 3.2. *Suppose $a \in L^2_\sigma$ satisfies* (3.7).
(i) *There exists a weak solution u, which is unique in case $n = 2$, such that*

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n+2}{4}}. \tag{3.13}$$

Furthermore, this weak solution satisfies

$$\|u(t)\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \textit{for all } 1 < q \le 2. \tag{3.14}$$

Proposition 3.1 implies $\|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n+2}{4}}$, so the existence of a weak solution with decay property (3.13) is deduced in exactly the same way as in [1], [16]. The assumption implies $a \in L^q_\sigma$ for all $1 < q \le 2$; so Proposition 3.1 and (3.13) together imply

$$\|e^{-tA}a\|_q \le C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})} \qquad \text{for all } 1 < q \le 2. \tag{3.15}$$

By using this, we can deduce assertion (3.14) in a standard manner.

To deal with strong solutions, note first that (IE) can be rewritten as

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds$$
$$= e^{-tA/2}u(t/2) - \int_{t/2}^t e^{-(t-s)A}P\nabla\cdot(u\otimes u)(s)ds.$$

We use this representation to prove

Theorem 3.3. *Let $a \in L^q_\sigma$ for all $1 < q < \infty$. Given $1 < p \leq 2$, there is a number $\eta_p > 0$ so that if $\|a\|_n \leq \eta_p$, a unique strong solution u exists for all $t \geq 0$, satisfying*

$$\begin{aligned}&\|u(t)\|_q \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\\&\|\nabla u(t)\|_q \leq Ct^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\end{aligned} \qquad \text{for all } p \leq q < \infty. \tag{3.16}$$

Theorem 3.3 is proved by following the argument given in [7], [8], [10]. Applying Proposition 3.1 and Theorem 3.3, we can deduce

Theorem 3.4. *Let $a \in L^1(\mathbb{R}^n_+) \cap L^q_\sigma$ for all $1 < q < \infty$ and satisfy (3.13). If a is small in L^n_σ, the strong solution u given in Theorem 3.4 satisfies*

$$\begin{aligned}&\|u(t)\|_q \leq C(1+t)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{q})}\\&\|\nabla u(t)\|_q \leq Ct^{-1-\frac{n}{2}(1-\frac{1}{q})}\end{aligned} \qquad \text{for all } 1 < q < \infty. \tag{3.17}$$

Our main result is

Theorem 3.5. (i) *For all $1 < q < \infty$, the strong solution u given in Theorem 3.5 satisfies*

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| & u^n(t) + 2U\partial_n E_t(\cdot)\int_{R^n_+} y_n a^n(y)dy\\ & -2U\partial_n F_t(\cdot)\cdot\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right)\Bigg\|_q = 0\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \Bigg\| & u'(t) + 2(\partial_n F_t(\cdot) - SU\partial_n E_t(\cdot))\int_{R^n_+} y_n a^n(y)dy\\ & +2\partial_n E_t(\cdot)\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right)\\ & +2SU\partial_n F_t(\cdot)\cdot\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty\int_{R^n_+} u^n u'(y,s)dyds\right)\Bigg\|_q = 0.\end{aligned} \tag{3.19}$$

(ii) *The weak solution u given in Theorem* 3.3 (ii) *satisfies* (3.18) *and* (3.19) *for* $1 < q \leq 2$.

We can apply Theorem 3.5 to characterizing flows with the lower bound of rates of energy decay. The result below extends a result of [12] to flows in the half-space.

Corollary 3.6. *The weak solution u given in Theorem* 3.3 (ii) *satisfies*

$$\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}} \qquad \textit{for large} \quad t > 0 \tag{3.20}$$

if and only if

$$\left(\int_{R^n_+} y_n a'(y)dy + \int_0^\infty \int_{R^n_+} (u^n u')(y,s)dyds, \quad \int_{R^n_+} y_n a^n(y)dy \right) \neq (0,0). \tag{3.21}$$

In [12] we proved that a flow u in $\mathbb{R}^n$ satisfies (3.20) if and only if

$$\left(\int y_j a^k(y)dy \right) \neq (0), \quad \text{or} \quad \left(\int_0^\infty \int (u^j u^k)(y,s)dyds \right) \neq c(\delta_{jk}). \tag{3.22}$$

We note that (3.22) involves all of the quantities $\int y_j a^k(y)dy$ and $\int_0^\infty \int (u^j u^k)(y,s)dyds$, while in (3.21) a distinguished role is played by the normal components a^n and u^n.

4 Proof of Theorem 2.1

We prove Theorem 2.1 by estimating linear and nonlinear terms separately.

Theorem 4.1. *Suppose a is solenoidal and satisfies*

$$\int (1+|y|)^m |a(y)|dy < \infty \tag{4.1}$$

for an integer $m \geq 1$. Then for $1 \leq q \leq \infty$,

$$\lim_{t\to\infty} t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| e^{-tA}a - \sum_{1\leq|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_x^\alpha E_t)(\cdot) \int y^\alpha a(y)dy \right\|_q = 0. \tag{4.2}$$

Proof. Since a is solenoidal and integrable, it satisfies (2.3). So,

$$(e^{-tA}a)(x) = \int [E_t(x-y) - E_t(x)]a(y)dy.$$

Applying Taylor's theorem to $E_t(x-y)$ and estimating the remainder, we can prove (4.2).

Consider next the nonlinear term

$$w(t) = (w^1(t), \cdots, w^n(t)) = -\int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds. \tag{4.3}$$

Theorem 4.2. *Under the assumption of Theorem* 2.1 (ii), *we have*

$$\left\| w^j(t) + \sum_{|\beta|+2p\le m-1} \frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(\cdot,t)\int_0^\infty\!\!\int s^p y^\beta (u^\ell u^k)(y,s)\,dyds\right\|_q = o(t^{-\frac{m}{2}-\frac{n}{2}(1-\frac{1}{q})}) \quad \text{as } t\to\infty, \tag{4.4}$$

for all $1\le q\le\infty$, $m=1,\cdots,n$ *and* $j=1,\cdots,n$.

Proof. By (4.3) and the definition of $F_{\ell,jk}$ given in Section 2, we have

$$w^j(t) = -\left(\int_0^{t/2}+\int_{t/2}^t\right)\int F_{\ell,jk}(x-y,t-s)(u^\ell u^k)(y,s)\,dyds \equiv J_1+J_2$$

and

$$\|J_2\|_q \le C\int_{t/2}^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{2q}^2\,ds = o(t^{-\frac{n}{2}-\frac{n}{2}(1-\frac{1}{q})}) \qquad \text{as } t\to\infty.$$

To estimate J_1, we write

$$\begin{aligned} J_1 &= -\int_0^{t/2}\!\!\int F_{\ell,jk}(x-y,t-s)(u^\ell u^k)(y,s)\,dyds \\ &= -\sum_{|\beta|+2p\le m-1}\frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_0^{t/2}\!\!\int s^p y^\beta(u^\ell u^k)(y,s)\,dyds \\ &\quad -\int_0^{t/2}\!\!\int R^m_{\ell,jk}(x,y,t,s)(u^\ell u^k)(y,s)\,dyds, \end{aligned}$$

to obtain

$$\begin{aligned} J_1 &+ \sum_{|\beta|+2p\le m-1}\frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_0^{\infty}\!\!\int s^p y^\beta(u^\ell u^k)(y,s)\,dyds \\ &= \sum_{|\beta|+2p\le m-1}\frac{(-1)^{|\beta|+p}}{p!\beta!}(\partial_t^p\partial_x^\beta F_{\ell,jk})(x,t)\int_{t/2}^{\infty}\!\!\int s^p y^\beta(u^\ell u^k)(y,s)\,dyds \\ &\qquad -\int_0^{t/2}\!\!\int R^m_{\ell,jk}(x,y,t,s)(u^\ell u^k)(y,s)\,dyds \\ &\equiv J_{11}+J_{12}. \end{aligned} \tag{4.5}$$

Recall (2.6), (2.7) and (2.12), *i.e.*, that

$$\begin{aligned} &\|(\partial_t^p\partial_x^\beta F_{\ell,jk})(\cdot,t)\|_q \le C_q t^{-\frac{1+|\beta|+2p}{2}-\frac{n}{2}(1-\frac{1}{q})}, \\ &\int s^p|y|^{|\beta|}|u(y,s)|^2dy \le C(1+s)^{-\frac{n-(|\beta|+2p)}{2}-1}. \end{aligned} \tag{4.6}$$

If $|\beta| + 2p \le m - 1$, each term of J_{11} behaves in L^q like $t^{-\frac{n+1}{2}-\frac{n}{2}(1-\frac{1}{q})}$ as $t \to \infty$; hence

$$t^{\frac{m}{2}+\frac{n}{2}(1-\frac{1}{q})}\|J_{11}\|_q \le Ct^{-\frac{n-m+1}{2}} \le Ct^{-\frac{1}{2}} \to 0 \qquad \text{as } t \to \infty.$$

So we need only estimate J_{12}, which is a finite linear combination of terms of the form

$$\int_0^1 \int_0^{t/2} \int (1-\tau)^{p-1} (\partial_t^p \partial_x^\beta F_{\ell,jk})(x, t - s\tau) s^p y^\beta (u^\ell u^k)(y,s) dy ds d\tau,$$

with $|\beta| + 2p = m$, or $m+1$, and $p > 0$; and

$$\int_0^1 \int_0^{t/2} \int (1-\theta)^{|\beta|-1} (\partial_x^\beta F_{\ell,jk})(x - y\theta, t - s) y^\beta (u^\ell u^k)(y,s) dy ds d\theta,$$

with $|\beta| = m$. Using (4.6) we easily see that all these terms behave in L^q as $o(t^{-\frac{m}{2}-\frac{n}{2}(1-\frac{1}{q})})$. This completes the proof of Theorem 4.2.

5 On Proof of Theorem 3.5 and Corollary 3.6

We first deduce Corollary 3.6 from Theorem 3.5. In view of (3.2), the functions $U\partial_n E_t$ and $U\partial_n F_t^j$, $j = 1, \cdots, n-1$, have the form $t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$, so

$$\|U\partial_n E_t\|_2^2 = C_1 t^{-\frac{n+2}{2}} > 0, \qquad \|U\partial_n F_t^j\|_2^2 = C_2 t^{-\frac{n+2}{2}} > 0.$$

We easily see that $U\partial_n E_t$ is an even function of x', and $U\partial_n F_t^j$ are odd functions of x'. Furthermore, let $j \le n-1$, $k \le n-1$ and $j \ne k$. Then $U\partial_n F_t^j$ is odd in x_j and even in x_k, while $U\partial_n F_t^k$ is odd in x_k and even in x_j. Therefore,

$$\begin{aligned} &(U\partial_n E_t, U\partial_n F_t^j) = 0, \qquad j = 1, \ldots, n-1, \\ &\|U\partial_n F_t^1\|_2^2 = \ldots = \|U\partial_n F_t^{n-1}\|_2^2, \\ &(U\partial_n F_t^j, U\partial_n F_t^k) = \delta_{jk} \|U\partial_n F_t^j\|_2^2, \end{aligned} \tag{5.1}$$

where $(\cdot,\cdot)$ is the inner product of $L^2(\mathbb{R}_+^n)$. ¿From (5.1) it follows that if we set

$$\alpha = 2\int_{R_+^n} y_n a^n(y) dy, \quad \beta = 2\int_{R_+^n} y_n a'(y) dy, \quad \gamma = 2\int_0^\infty \int_{R_+^n} (u^n u')(y,s) dy ds,$$

then

$$\begin{aligned} \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2^2 &= \|U\partial_n E_t\|_2^2 \alpha^2 + \|U\partial_n F_t^1\|_2^2 |\beta+\gamma|^2 \\ &= Ct^{-\frac{n+2}{2}}. \end{aligned} \tag{5.2}$$

We shall apply (5.2) to the proof of Corollary 3.6. Firstly, suppose that $(\beta+\gamma, \alpha) \ne (0,0)$. Then (5.2) implies

$$\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 = Ct^{-\frac{n+2}{4}} > 0 \qquad \text{for all } t > 0;$$

so (3.18) yields, for large $t > 0$,

$$\begin{aligned}\|u^n(t)\|_2 &\geq \|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\quad - \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geq ct^{-\frac{n+2}{4}}.\end{aligned}$$

Secondly, suppose that $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ for large $t > 0$. Then (3.18) implies

$$\begin{aligned}&\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\qquad \geq \|u^n(t)\|_2 - \|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\qquad \geq ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) > 0\end{aligned}$$

for large $t > 0$, and so we conclude that $(\beta+\gamma,\alpha) \neq (0,0)$. Suppose finally that

$$\liminf_{t\to\infty} t^{\frac{n+2}{4}}\|u^n(t)\|_2 = 0 \qquad \text{and} \qquad \|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}. \tag{5.3}$$

In this case we invoke

$$\begin{aligned}C &= t^{\frac{n+2}{4}}\|U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 \\ &\leq t^{\frac{n+2}{4}}\|u^n(t) + U\partial_n E_t(\cdot)\alpha - U\partial_n F_t(\cdot)\cdot(\beta+\gamma)\|_2 + t^{\frac{n+2}{4}}\|u^n(t)\|_2.\end{aligned}$$

Passing to the limit as $t \to \infty$ and applying (3.18) and (5.3) gives $C = 0$, since

$$\liminf_{t\to\infty}[f(t) + g(t)] = \lim_{t\to\infty} f(t) + \liminf_{t\to\infty} g(t).$$

This implies that $(\beta+\gamma,\alpha) = (0,0)$, so (3.18) and (3.19) together yield

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|u(t)\|_2 = 0,$$

contradicting the assumption (5.3). Hence $\|u(t)\|_2 \geq ct^{-\frac{n+2}{4}}$ implies $\|u^n(t)\|_2 \geq ct^{-\frac{n+2}{4}}$, and so we get $(\beta+\gamma,\alpha) \neq (0,0)$. This completes the proof of Corollary 3.6.

The proof of Theorem 3.5 is so long and complicated that we cannot present it here; for the details the reader is referred to [4]. Here we write a lemma which plays a crucial role in expanding linear and nonlinear terms of (IE).

Lemma 5.1. *Let $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n$, and consider the function*

$$K(x,y,t) = t^{-\frac{n+1}{2}}K^0(xt^{-\frac{1}{2}}, yt^{-\frac{1}{2}})$$

where $K^0(\xi,\eta)$ is smooth and satisfies

$$\|\nabla^m_{\xi,\eta}K^0(\cdot,\eta)\|_q \leq C_{m,q}$$

for all $m = 0, 1, 2, \ldots$, *and some* $1 < q \leq \infty$. *If we set*

$$\begin{aligned} I_*(x,t) &= \int_0^{t/2} \int K(x,y,t-s)(u\otimes u)_*(y,s)dyds \\ I^*(x,t) &= \int_0^{t/2} \int K(x,y,t-s)(u\otimes u)^*(y,s)dyds, \end{aligned} \tag{5.4}$$

with u *the strong solution treated in the main result, then*

$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \left\| I_*(\cdot,t) - 2K(\cdot,0,t)\int_0^\infty \int_{R^n_+} (u\otimes u)(y,s)dyds \right\|_q = 0$$
$$\lim_{t\to\infty} t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})} \|I^*(\cdot,t)\|_q = 0.$$

Here f^* *is the odd extension and* f_* *is the even extension, both in* x_n*, of a function* $f(x', x_n)$ *defined on* $\mathbb{R}^n_+$.

Using the operators as listed in Section 3, we decompose each term on the right-hand side of (IE) into the terms of the form (5.4), and then apply Lemma 5.1, to complete the proof of Theorem 3.5. The way of decomposition is similar to that given in Section 4, but the resulting terms look much more complicated. The detailed arguments are given in [4].

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Remark on the L^2 decay for weak solution to equations of non-Newtonian incompressible fluids in the whole space (II)

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1 Introduction

There are fluids that do not obey the simple relationship between shear stress and shear strain rate given by the following equation for a Newtonian fluid

$$T = -\pi\delta_{ij} + 2\mu e_{ij}. \tag{1.1}$$

These fluids have been given the general name non-Newtonian fluids. Many common fluids are non-Newtonian: paints, solutions of various polymers, food products such as apple sauce and ketchup, emulsions of water in oil or oil in water and suspensions of various solids and fibers in a liquid paper pulp or coal slurries and the drilling mud used in well drilling. We shall consider the fluids which exhibit the property to shear thin or shear thicken-the power law fluids. The equations governing the unsteady motion of these fluids have been previously investigated by many authors, e.g. [6]. We are interested in the asymptotic behaviour of the power law fluids. The Cauchy problem of the power-law incompressible fluids in R^3 is described by the system of equations

$$\begin{aligned} \operatorname{div} u &= 0, \\ u_t + (u\cdot\nabla)u &= \operatorname{div} T, \\ u(.,0) &= u_0, \end{aligned} \tag{1.2}$$

$u = (u_1, \ldots, u_3)$ represents the velocity field, T is the stress tensor, u_0 is the initial

value of the velocity. The stress tensor is decomposed as

$$T_{ij} = -\pi\delta_{ij} + \tau^v_{ij}, \tag{1.3}$$

where π is the pressure, δ_{ij} is the Kronecker delta, τ^v is the viscous part of the stress. We will assume the stress tensor τ^v of the form

$$\tau^v = \tau(\mathbf{D}u) \tag{1.4}$$

with τ a nonlinear tensor function , where the components of the symmetric deformation velocity tensor are given by

$$\mathbf{D}u_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{1.5}$$

We consider the following growth-conditions

$$\begin{aligned} |\tau_{ij}(\mathbf{D}u)| \le C_1(|\mathbf{Du}| + |\mathbf{Du}|^{p-1}), \quad C_1 > 0,\ p \ge 2, \\ |\tau_{ij}(\mathbf{D}u)| \le C_1|\mathbf{Du}|^{p-1}, \quad 1 < p < 2, \end{aligned} \tag{1.6}$$

as well as the strong coercivity condition

$$\tau_{ij}(\mathbf{D}u)\mathbf{D}u_{ij} \ge C_2(|\mathbf{Du}|^p + |\mathbf{Du}|^2),\ 1 < p < \infty,\ C_2 > 0, \tag{1.7}$$

$(|\mathbf{D}| = (\mathbf{D}u_{ij}\mathbf{D}u_{ij})^{\frac{1}{2}})$.

The existence of a weak solution for $p > \frac{3n}{n+2}$ and uniqueness and regularity for $p \ge 1 + \frac{2n}{n+2}$ $n = 2, 3$ were proved by [6], 1996.

There is an extensive literature on decay on the Navier-Stokes equations. We recall some of them. First result on decay of strong solutions with small initial data was given by Kato [6]. The decay in L^2 norm of weak solution with arbitrary data was proved by Schonbek [14] assuming $L^1 \cap L^2$ integrability of the initial data. Further Wiegner [21] and Kajikiya, Miyakawa [5] improved decay replacing assumption on L^1 integrabilty to asumption on L^p integrability see [4] resp. that initial data are also initial data of linear heat equation see [15]. There are also many results on L^1 decay and decay in Hardy spaces given by Miyakawa see [9]. There are a lot of results of decay in an exterior domain, we can mention e.g. work of Masuda and Heywood see [4],[7]. Concerning lower bounds we can mention works of Schonbek see [17]-[19]. There is intensive investigation of the boundedness and decay of moments see [20],[2].

The aim of this paper is to investigate the problem of shear thickenning ($p > 2$) when the stress tensor has the following form

$$\tau^v_{ij} = (\mu_0 + \mu_1|\mathbf{D}u|^{p-2})\mathbf{D}u_{ij}. \tag{1.8}$$

We shall not assume u_0 to be small or belong to some L_p space but u_0 is also initial data of the linear heat equation. We see that the decay results depend solely

on the decay properties of the solution of the heat equation. Finally, we deal with lower bounds of rates of decay of power law fluids. We consider the case when the average of $\int_{R^n} u_0 dx$ is nonzero.

We denote by $C(R^n)$ and $C^k(R^N)$ ($k \in N$ or $k = \infty$) the space of real continuous functions on R^n and the space of k-times continuously differentiable functions on R^n, respectively. The space of real C^∞ functions on R^n with a compact support in R^n is denoted by $D(R^n)$ and its dual by $D'(R^n)$.

Denote $I = (0,T)$ with $T > 0$, $Q_T = R^n \times I$. The standard notation is used for both scalar ($u : R^n \mapsto R$ or $Q_t \mapsto R$) and vector-valued functions ($u : R^n \mapsto^n$ or $Q_T \mapsto R^n$). The Lebesgue spaces of scalar and vector-valued functions are denoted by $L^q(R^n)$ and $L^q(R^n)^n$, respectively, $q \in [1,\infty)$ with the standard norm $\|\cdot\|_q$. By $L^{q'}$ we denote the dual space to L^q, where $1/q + 1/q' = 1$. The Sobolev spaces $W^{m,p}(R^n)$ and $W^{m,p}(R^n)^n$ are the sets of all measurable functions, for which the functions and all their generalized derivatives up to the order m belong to $L^p(R^n)$ and $L^p(R^n)^n$, respectively. The spaces are equipped with the standard norms, semi-norms are denoted by $\|\cdot\|_{m,q}$ and $|\cdot|_{m,q}$ (for detailed descriptions see [1]). By $W^{-m,q'}(R^n)$, $W^{-m,q'}(R^n)^n$ we denote the dual space to $W^{m,q}(R^n)$, $W^{m,q}(R^n)^n$, respectively.

Let s be a noninteger positive number, $s = [s] + \{s\}$, where $[s]$ is the integer and $\{s\}$ the fractional part of s and let $1 \le p < \infty$ then the Slobodeckij spaces $W^{s,p}(R^n)((W^{s,p}(R^n)^n)$ are subsets of the Sobolev spaces $W^{[s],p}(R^n)$ $(W^{[s],p}(R^n)^n)$, where

$$\|u\|_{s,p} = \|u\|_{[s],p} + \sum_{|\alpha|=[s]} \int_{R^n \times R^n} \Big(\frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\{s\}p}} dxdy\Big)^{\frac{1}{p}} < \infty.$$

Let X be a Banach space with the norm $\|.\|_X$. The Bochner spaces $L^q(I,X)$ consist of all measurable function $u : I \to X$ the norm of which $\|u\|^q_{L^q(I,X)} := \int_0^T \|u(t)\|^q_X dt$ is finite. For $q = \infty$, the norm is $\|u\|_{L^\infty(I,X)} = ess\sup_{t\in I} \|u(t)\|_X$.

We denote by $H = \{\varphi \in L^2(R^3)^3; \mathrm{div}\varphi = 0\}$ and by $V_p = \{\varphi \in D'(R^3)^3, \nabla\varphi \in L^p(R^3),\ \mathrm{div}\ \varphi = 0\}$.

Lemma 1.1 *(generalized Korn inequality) Let $\varphi \in W^{1,q}(R^3)^3 \cap W^{1,2}(R^3)^3$, $q > 1$. Then*

$$\Big(\int_{R^3} |\mathbf{D}\varphi|^q dx\Big) \ge k_q |\varphi|^q_{1,q}, \tag{1.9}$$

where $k_q > 0$.

Proof: see [11]

2 Formulation of the problem

We consider the initial data u_0 belong to the set B_α, with $\alpha \geq 0$ if the following condition is satisfied:

$$||a(t)||_2^2 \leq (1+t)^{-\alpha} \text{ for some } C > 0,$$

where $a(t)$ denotes the semigroup solution of the heat system $\frac{\partial a}{\partial t} - \Delta a = 0$ on $R^+ \times R^n$ with initial value u_0.

Remarks:
(i) It is easily seen that $||a(t)||_2 \to 0$ for $t \to \infty$,
(ii)If $A = -\Delta$, we may easily derive the estimate

$$||e^{-tA}u||_q \leq t^{-(n/r-n/q)/2}||u||_r, \text{ for } 1 \leq r \leq q \leq \infty.$$

If $u_0 \in L_p(R^n)^n$, with $1 \leq p < 2$ then $u_0 \in B_\alpha$, where $\alpha = n((1/p) - 1/2)$.

We define a weak solution of (1.2),(1.3),(1.8) and formulate the main result.

Definition 2.1. Let $u_0 \in W^{1,2}(R^3)^3 \cap H$ and $p \geq \frac{9}{5}$. Then a function u where

$$u \in L^p(I; V_p) \cap L^\infty(I; H) \tag{2.1}$$

is called a weak solution of the problem (1.2),(1.3),(1.8) if

$$\begin{aligned} &-\int_{Q_T} u_i \frac{\partial \varphi_i}{\partial t} dxdt + \int_{Q_T} u_j \frac{\partial u_i}{\partial x_j} \varphi_i dxdt \\ &+ \int_{Q_T} \tau_{ij}(\mathbf{D}u)\mathbf{D}\varphi_{ij} dxdt = \int_{R^3} u_{0i}\varphi_i(0) dx. \end{aligned} \tag{2.2}$$

is satisfied for every $\varphi \in C^1(I, \mathcal{D}_0(R^n)^n), \varphi(T) = 0$ and $\mathrm{div}\varphi = 0$.

Theorem 2.1 *Let $u_0 \in H \cap W^{1,2}(R^3)^3$, $p \geq 1$. Then there exists at least one weak solution of the problem (1.2),(1.3),(1.8) in the sense of Definition 2.1.*

Proof: see [11].

We present an argument to establish L^2 decay of solutions to the Cauchy problem for 3-dimensional incompressible non-Newtonian fluids.

Theorem 2.2 *(Main Theorem)*
Let $u \in L^\infty(I, V_p(R^3)^3) \cap L^\infty(I, H)^3) \cap L^2(I, W^{1,2}(R^3)^3)$ with u vanishing at infinity. Suppose that u and π satisfy

$$\begin{aligned} u_t + (u \cdot \nabla)u - div\tau^\nu + \nabla\pi = 0, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \end{aligned} \tag{2.3}$$

with $u_0 \in L^2(R^3)^3 \cap B_{\alpha_0}$, $p > 3$ *then*

$$||u(.,t)||_2^2 \leq C(t+1)^{-\bar{\gamma}} \tag{2.4}$$

where c depends only on L^2 *norm of* $u_0, \bar{\gamma} = min\{\alpha_0, \frac{1}{2}n + 1\}$ *and the following energy inequality is satisfied*

$$\frac{d}{dt}\int_{R^3} |u|^2 + c_1 \int_{R^3} |\nabla u|^2 + c_2 \int_{R^3} |\nabla u|^p \leq 0.$$

Moreover,

$$||u(.,t)||_2 \to 0 \text{ for } t \to \infty. \tag{2.5}$$

Finally, the solution u is asymptotically equivalent to the solution of the heat system with the same data in the sense that

$$||u(t) - a(t)||_2^2 \to 0. \tag{2.6}$$

3 Technical Lemma

In this section we prove technical lemma which we need to prove the main theorem.

Decompose the frequency domain into two time-dependent subdomains yields a first-order differential inequality for the spatial L^2 norm of the Fourier transformation of the solution. The time dependent subdomains are n-dimensional sphere $S(t)$, centered at the origin with an appropriate time-dependent radius and its complement. As in [14], L^2 decay will depend on an estimate of the Fourier transforms of u for the frequency value $\xi \in S$.

Lemma 3.1 *Let the assumptions of Main theorem be satisfied. Then the following inequality holds*

$$\begin{gathered} ||u(t)||_2^2 exp(\int_0^t g^2(s)ds) \leq ||u_0||_2^2 + c\int_0^t \frac{d}{ds}(exp(\int_0^t g^2(r)dr)) \\ \times\{||a(t)||_2^2 + g^{n+2}(s) + g^{n+2}(s)(\int_0^s ||u(r)||_2^2)^2\}ds, \end{gathered} \tag{3.1}$$

where $n = 3$.

Proof:

First, we write the energy equality

$$\frac{d}{dt}||u||^2 + \mu_0||\nabla u||^2 + c\mu_1 \int_{R^3} |\mathbf{D}u|^p dx = 0. \tag{3.2}$$

¿From the previous paper we know that

$$\int_0^t ||\mathbf{D}u||_{p-1}^{p-1} d\tau \leq c \tag{3.3}$$

for $p > 3$. Using the Plancheler theorem we get (F denotes the Fourier transformation). For the technical reason let $\mu_0 = \mu_1 = 1$.

$$\begin{aligned} &\tfrac{d}{dt}\|F(u)\|^2 + \|\mathbf{D}u\|_p^p dx \le -\int_{R^3} |\xi|^2 |F(u)|^2 d\xi \\ &\le -c(\int_{|\xi|\le g(t)} + \int_{|\xi|\ge g(t)}) |\xi|^2 |F(u)|^2 d\xi \\ &\le -c\int_{|\xi|\le g(t)} |\xi|^2 |F(u)|^2 - cg^2(t) \int_{|\xi|\ge g(t)} |F(u)|^2 d\xi \\ &\le c\int_{|\xi|\le g(t)} (g^2(t) - |\xi|^2)|F(u)|^2 - cg^2(t) \int_{R^3} |F(u)|^2 d\xi \le \\ &\le c\int_{|\xi|\le g(t)} g^2(t)|F(u)|^2 d\xi - cg^2(t)\int_{R^3} |F(u)|^2 d\xi. \end{aligned} \tag{3.4}$$

Then

$$\frac{d}{dt}\|F(u)\|^2 + cg^2(t)\int_{R^3} |F(u)|^2 d\xi + \int_{R^3} |\mathbf{D}u|^p dx \le \int_{|\xi|\le g(t)} g^2(t)|F(u)|^2 d\xi. \tag{3.5}$$

Taking the Fourier transform of $(1.2)_2$ we get

$$\widehat{u}_t + |\xi|^2\widehat{u} = G(\xi, t) \tag{3.6}$$

where

$$\begin{gathered} G(\xi, t) = -F(u\cdot\nabla u) - \xi F(\pi) - \xi F(\widetilde{\tau}_{ij}) \\ (\widetilde{\tau_{ij}} = |\mathbf{D}u|^{p-2}\mathbf{D}u_{ij}). \end{gathered} \tag{3.7}$$

Multiplying (3.1) by $e^{|\xi|^2 t}$ we have

$$\frac{d}{dt}[e^{|\xi|^2 t}\widehat{u}] = e^{|\xi|^2 t}G(\xi, t). \tag{3.8}$$

It follows that

$$|\widehat{u}(\xi, t)| \le e^{-|\xi|^2 t}|\widehat{u}_0(\xi)| + \int_0^t e^{-|\xi|^2(t-s)}|G(\xi, s)|ds. \tag{3.9}$$

Now,

$$\begin{gathered} \int_{|\xi|\le g(t)} |F(u)|^2 d\xi \le \int_{|\xi|\le g(t)} e^{-2|\xi|^2 t}|F(u_0)|^2 d\xi + \\ c\int_{|\xi|\le g(t)} (\int_0^t e^{-|\xi|^2(t-s)} G(\xi, s))^2 d\xi. \end{gathered} \tag{3.10}$$

We are dealing with the first term of the right-hand side

$$\begin{gathered} \int_{|\xi|\le g(t)} e^{-2|\xi|^2 t}|F(u_0)|^2 d\xi \le \int_{R^3} |F(u_0)|^2 e^{-2|\xi|^2 t} \le \\ \le \|u_0\|_2^2 = \|a\|_2^2. \end{gathered} \tag{3.11}$$

We estimate the term G.

$$\begin{aligned} &\int_{|\xi|\le g(t)} (\int_s^t e^{-|\xi|^2(t-s)} F(\nabla\cdot|\mathbf{D}u|^{p-1})ds)^2 d\xi \\ &\le c\int_{|\xi|\le g(t)} |\xi|^2 (\int_s^t \|\mathbf{D}u\|_{p-1}^{p-1} dx ds)^2 d\xi \le g^{n+2}, \end{aligned} \tag{3.12}$$

$$\int_{|\xi|\le g(t)} (\int_0^t e^{-|\xi|^2(t-s)} F(u\cdot\nabla u)ds)^2 d\xi \le cg^{n+2}(t)(\int_0^t \|u\|^2 ds)^2. \tag{3.13}$$

Finally, we deal with the term with pressure. Since

$$\Delta\pi = \frac{\partial^2}{\partial x_i \partial x_j}(|\mathbf{D}u|^{p-2}\mathbf{D}u - u \otimes u) \tag{3.14}$$

and applying the Fourier transformation we get

$$|\xi|^2 F(u) = \xi_i \xi_j (F(|\mathbf{D}u|^{p-2}\mathbf{D}u) - F(u \otimes u)). \tag{3.15}$$

Then

$$\int_{|\xi| \leq g(t)} (\int_0^t e^{-|\xi|^2(t-s)} F(\nabla\pi) ds)^2 d\xi \leq c g^{n+2}(1 + \int_0^t \|u\|_2^2 ds)^2). \tag{3.16}$$

which gives us (3.1).

4 Proof of Main theorem

Let asssume that

$$\|u(s)\|_2^2 \leq c(1+t)^{-\beta},$$

where $\beta \geq 0$. Take $g^2(t) = \alpha(1+t)^{-1}$, with α suitable large and positive. Then

$$exp(\int_0^t g^2(s)ds) = (1+t)^{\alpha}.$$

Applying (16) we get

$$\begin{aligned} &\|u(t)\|^2(1+t)^{\alpha} \leq \|u_0\|_2^2 + C\int_0^t (1+s)^{\alpha-1}\{\|a_(s)\|^2 + \\ &(1+s)^{-\frac{n+2}{2}} + (1+s)^{-\frac{n-2}{2}-2\beta}\}ds \leq c(1+t)^{-\beta_1+\alpha}. \end{aligned} \tag{4.1}$$

with $\beta_1 = min\{\frac{n-2}{2} + 2\beta, \alpha_0\}$. Then

$$\|u(t)\|_2^2 \leq (1+t)^{-\beta_1}. \tag{4.2}$$

If $\beta_1 = \alpha_0$ then the proof is completely otherwise, we consider a new exponent β_1. Repeating this process several times we get

$$\|u(t)\|_2^2 \leq (1+t)^{-\alpha_0}, \tag{4.3}$$

if $\alpha_0 \leq 1$. If $\alpha_0 > 1$, we obtain after many iteration $\tilde{\beta}$ of the form $\tilde{\beta} = 1 + \epsilon$ with $\epsilon > 0$. Now, since

$$\int_0^s \|u(r)\|_2^2 dr \leq C$$

which is independent of s and applying again Lemma 3.1 we get

$$\|u(t)\|_2^2(t+1)^{\alpha} \leq C + c(t+1)^{\alpha-\alpha_0} + (t+1)^{\alpha-(\frac{1}{2}n+1)}, \tag{4.4}$$

hence

$$\|u(t)\|_2^2 \le C(t+1)^{\tilde{\gamma}} \tag{4.5}$$

for $\tilde{\gamma} = min\{\alpha_0, \frac{1}{2}n + 1\}$.

Now, applying again Lemma 3.1 with $g(t) = \alpha t^{-1}$, $\alpha > 0$ and using that $\|a(t)\|_2 \to 0$, $\|u(r)\|_2 \le \|a(r)\|_2$ we obtain (2.5).

Finally, we are dealing with (2.6). We denote $v(t) = u(t) - a(t)$. Using energy inequality (3.2) and

$$\frac{d}{dt}\|a\|_2^2 + \|\nabla a\|_2^2 = 0, \tag{4.6}$$

we get

$$\frac{d}{dt}\|v\|_2^2 + \|\nabla v\|_2^2 \le 0. \tag{4.7}$$

Similarly as in Lemma 3.1 we get

$$\|F(v)\|_2^2 \le g^{n+2}\Big(1 + (\int_0^t \|u(r)\|_2^2\, dr)^2\Big), \tag{4.8}$$

and

$$\begin{aligned} &\|v(t)\|_2^2 exp(\textstyle\int_0^t g^2(s)ds) \le c \int_0^t \frac{d}{ds}(exp(\int_0^t g^2(r)dr)) \\ &\times\{g^{n+2}(s) + g^{n+2}(s)(\textstyle\int_0^s \|u(r)\|_2^2)^2\}ds. \end{aligned} \tag{4.9}$$

As in the previous parts, applying Lemma 3.1 with $g^2(t) = \alpha(t+1)^{-1}$ and the fact that $\|u(t)\|_2^2 \le C(t+1)^{\tilde{\gamma}}$ we get (2.6).

5 Lower bounds

In this section we obtain the lower bounds for the rates of decay for the solution to the 3-dimensional power law fluids. We denote by $S_\alpha^\delta = \{u : |\hat{u}(\xi)| \ge \alpha \text{ for } |\xi| \le \delta\}$.

Let v be a solution to the heat equation

$$\begin{aligned} v_t &= v_{xx} \\ v(x,0) &= u_0(x). \end{aligned} \tag{5.1}$$

Lemma 5.1 *Let $u_0 \in L^2(R^3) \cap S_\alpha^\delta$ for some $\alpha, \delta > 0$. If v is a solution of the heat equation (5.1) with data u_0 then*

$$\int_{R^3} |v|^2 dx \ge \int_{R^3} c(t+1)^{-\frac{3}{2}} \tag{5.2}$$

where c depending only on α and δ.

Proof: see [17].

Lemma 5.2 *Let $u_0 \in L^2(R^3)$. If v is a solution of (5.1) then*

$$|\nabla v|_\infty \le ct^{-\frac{5}{4}}, \tag{5.3}$$

where the constant c depends only on the L^2 norm of u_0.

Proof: see [17].

Theorem 5.1 *Let $u_0 \in L^1 \cap H$, $divu_0 = 0$. Then there exists a weak solution satisfying*

$$|u(.,t)|_2^2 \ge c(t+1)^{-\frac{3}{2}}, \tag{5.4}$$

with c depending only on L^1 and L^2 norms of u_0.

Proof: Let $w = u - v$ where v is the solution of the linear heat equation (5.1) with data u_0. We write the equation for w

$$w_t = \Delta w - (u\nabla u + \nabla p) - \text{ div } (|\mathbf{D}u|^{p-2}\mathbf{D}u) \tag{5.5}$$

Multiplying the equation (5.5) by w and integrating in space yields

$$\begin{array}{l} \frac{d}{dt}\int_{R^3}\frac{|w|^2}{2} = -\int_{R^3}|\nabla w|^2 - \int_{R^3}(u-v)u\nabla u - \\ \int_{R^3} w\nabla p + \int_{R^3}(u-v) \text{ div } (|\mathbf{D}u|^{p-2}\mathbf{D}u). \end{array} \tag{5.6}$$

Since div $w = 0$, div $u = 0$, $\int_{R^3} u\cdot u\nabla u = 0$. Hence

$$\begin{array}{l} \frac{d}{dt}\int_{R^3}|w|^2dx = -2\int_{R^3}|\nabla w|^2 - 2\sum_j\int_{R^3}\sum_i u^j\nabla u^j - \\ -\int_{R^3}(u-v) \text{ div } (|\mathbf{D}u|^{p-2}\mathbf{D}u) \\ \le -2\int_{R^3}|\nabla w|^2 + c|\nabla v|_\infty\int_{R^3}|u|^2dx + c|\nabla v|_\infty\int_{R^3}(|\mathbf{D}u|^{p-2}\mathbf{D}u)dx. \end{array} \tag{5.7}$$

Let $S(t) = \{\xi : |\xi| \le \left(\frac{3}{2(t+1)}\right)^{\frac{1}{2}}\}$. Applying Theorem 2.2 we get

$$\begin{array}{rcl} \frac{d}{dt}[(t+1)^3\int_{R^3}|w|^2] & \le & 3(t+1)^2\int_S|\widehat{w}|^2dx + c(t+1)^3|\nabla v|_\infty\int_{R^3}|u|^2dx + \\ & & +c(t+1)^3|\nabla v|_\infty\int_{R^3}(|\mathbf{D}u|^{p-2}\mathbf{D}u)dx. \end{array} \tag{5.8}$$

Using Lemma 5.2 and Theorem 2.2 we obtain

$$\frac{d}{dt}[(t+1)^3\int_{R^3}|w|^2] \le 3(t+1)^2\int_S|\widehat{w}|^2dx + c(t+1)^{\frac{3}{2}}t^{-\frac{5}{4}} + c(t+1)^3t^{-\frac{5}{4}}. \tag{5.9}$$

We need the estimate on $\widehat{w}$ for low frequencies. The difference $\widehat{w}$ satisfies

$$\left.\begin{array}{l} \widehat{w}_t + |\xi|^2\widehat{w} = -u\widehat{\nabla}u - \widehat{\nabla}\pi - F(\text{ div } (|\mathbf{D}u|^{p-2}\mathbf{D}u)) = H \\ \widehat{w}(\xi,0) = 0. \end{array}\right\} \tag{5.10}$$

Hence by [8] (Lemma 3.2) and Theorem 2.2 we have

$$|H| \le c|\xi|||u||_2^2 \le c|\xi|\frac{1}{(s+1)^{3/2}}. \tag{5.11}$$

Since $|\xi| \le \left(\frac{3}{2(t+1)}\right)^{1/2}$, $\xi \in S(t)$, we get

$$|\widehat{w}(\xi,t)| \le \frac{c}{(t+1)^{1/2}}. \tag{5.12}$$

Altogether give us

$$\frac{d}{dt}((t+1)^3 \int_{R^3} |w|^2) \le C((t+1)^{2-1/2} \int_S d\xi + c(t+1)^{\frac{3}{2}} t^{-\frac{5}{4}} + c(t+1)^3 t^{-\frac{5}{4}}). \tag{5.13}$$

Integrating over $[1,t]$ we obtain

$$\begin{aligned}&(t+1)^3 \textstyle\int_{R^3} |w|^2 \le Ct + 2c\int_0^t (s+1)^{3/2}(s+1)^{-\frac{5}{4}} ds + (s+1)^3(s+1)^{-\frac{5}{4}} + \\ &\textstyle\int_{R^3} |w(1)|^2 dx \le Ct + 2c(t+1)^{5/4} + 2c(t+1)^{\frac{11}{4}} + \int_{R^3} |w(1)|^2 dx.\end{aligned} \tag{5.14}$$

Since

$$\int_{R^3} |w(1)|^2 dx \le 2\int_{R^3} |u_0|^2 dx \tag{5.15}$$

then

$$\begin{aligned}&\textstyle\int_{R^3} |w|^2 dx \le C_0[(t+1)^{-2} + (t+1)^{-\frac{7}{4}} + (t+1)^{-3} + (t+1)^{-\frac{1}{4}}] \\ &\le c_0[(t+1)^{-3/2-1/4} + (t+1)^{-\frac{1}{4}}].\end{aligned} \tag{5.16}$$

Applying

$$\|u\|_2^2 \ge \|v\|_2^2 - \|w\|_2^2 \tag{5.17}$$

and Lemma 5.1 we get

$$\|u\|_2 \ge c(t+1)^{-3/4}. \tag{5.18}$$

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Navier-Stokes Simulations of Vortex Flows

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1 Introduction

Vortices play a dominant role in many flow problems, in hydrodynamics as well as in aerodynamics. Vortices can influence the main flow substantially, as for example, the dangerous spiraling of vortices in diffusers of large water turbines or the lee-side vortices on delta wings. The notion of slender vortices will be introduced first in the frame of Prandtl's lifting-line theory. Simplified model equations will be used to explain the flow motion in the cores of the starting and the tip vortices. The simulation of vortex flows facilitated with numerical integration will be explained for time dependent flows in comparison with experimental data. The results show, that the motion of the vortices and also their inner structure can accurately be simulated with numerical solutions of the Navier-Stokes equations on high-performance computers.

In his celebrated paper on the lifting line theory Prandtl introduced the concept of slender vortices into aerodynamics [1]. He assumed that the flow around a wing can be simulated with the aid of rectilinear vortex filaments, superimposed on a uniform velocity field. The vortex system he used consists of the bound vortex, replacing the wing, the tip vortices, and the starting vortex. The tip vortices and the starting vortex are long or better slender with the ratio of the diameter D to the length of the vortex L much smaller than unity. He further assumed that in the vortex core the flow obeys rigid body rotation, but does not have any variation in the axial direction. Outside of the core the flow is given by a potential vortex. This model has served as a very useful purpose over decades to simulate the lift and also the induced drag of a wing of finite span.

In contrast to Prandtl's mathematical model for the vortex system of a wing, in the case of real flows there is axial flow in the core. This was shown, for example, in ref-

erences [2], [3], and [4] in an experiment with a starting vortex. In that experiment a flap was hinged to one side wall of a glas container with rectangular cross-section; the flap extended all the way to the other two side walls. When the container was filled with water a starting vortex could be generated by rotating the flap; the flow was visualized by injecting dye of different colors at the trailing edge of the flap. The experiment showed strong axial flow motion, setting in at both ends of the vortex with radial flow motion on the wall which then turned into axial flow in the core of the vortex, and because of its counter-current nature it destroyed the prevailing slender flow structure of the vortex.

Similar to the starting vortex, the tip vortices experience changes in their cores, too. This can even be recognized in the sky at favorable weather conditions, as the vapor tails of aircraft visualize their tip vortices. The two counter-rotating, outstandingly slender vortices tend to become unstable and interact with each other. They form vortex cells in the wake of an aircraft, visible in the sky, weather permitting. The destabilization of the flow was investigated theoretically years ago by S. C. Crow in [5] and is since then referred to as the Crow-instability. His findings were later confirmed with an experimental investigation of T. Leweke and C. H. K. Williamson, reported in [6] and [7].

Upon a recommendation of R. Rautmann the major part of this paper follows some of the developments of reference [8].

2 Qualitative analysis of the flow in slender vortices

The axial flow behavior in slender vortices can qualitatively be analyzed in the following manner. The slenderness condition already mentioned, in the form $D/L << 1$ can be redefined with an order-of-magnitude consideration. The ratio $D/L << 1$ can be replaced by the projection of streamlines:

$$dr/dx = v/u << 1 \tag{2.1}$$

In equ. (2.1) the quantities r and x stand for the radial and axial coordinates, respectively, and v and u are the corresponding velocity components. With the aid of the continuity equation and the x momentum equation

$$u_x + (rv)_r/r = 0 \tag{2.2}$$

$$uu_x + vu_r + p_x/\rho = 0 \tag{2.3}$$

the slenderness condition can be rewritten in the following form

$$dr/dx = v/u = (1/r)\int_r^\infty (r'p_x/\rho u^2)dr' << 1 \quad . \tag{2.4}$$

¿From equ. (2.4) it follows, that the slenderness condition, equ. (2.1) cannot be satisfied, if

$$p_x > 0, \quad u \rightarrow 0, \tag{2.5}$$

and consequently the flow structure in the core has to change. For the axial flow motion in the vortex core three types can be distinguished, namely uniform, jet-like and wake-like axial flow. The jet-like axial flow is characterized by an velocity overshoot along the axis, and the wake-like flow by a velocity defect in comparison to the flow outside of the core. Equs. (2.4) and (2.5) show, that wake-like axial flow profiles with positive axial pressure gradient p_x tend to increase the slope of the stream-lines, so that eventually the slenderness conditions can no longer be satisfied. Axial flows with jet-like profiles also tend to increase the slope of the stream-lines, when the axial pressure gradient p_x is positive, but the decrease is slower. If the radial profile of the axial pressure gradient is negative to the extent that the integral in equ. (2.5) is negative, the slope of the stream-lines is even decreased and the vortex remains slender.

In turn, the axial flow can strongly be influenced by the azimuthal velocity distribution. This can be demonstrated with the aid of the radial and azimuthal momentum equations:

$$\begin{aligned} w^2/r &= p_r/\rho && (2.6) \\ uw_x + vw_r + vw/r &= 0 && (2.7) \end{aligned}$$

By differentiating the radial momentum equation (2.6) with respect to x, by inserting the azymuthal momentum equation (2.7) into the x-derivative of p_r, and by integrating the resulting relation with respect to r, there is obtained:

$$p_x(x,r) = p_x(x,\infty) + 2\rho \int_r^\infty (v/u)(w/r'^2)(r'w)_{r'} dr' \quad . \tag{2.8}$$

The above relation shows, that the azimuthal velocity can induce an axial pressure gradient, even if

$$p_x(x,\infty) = 0 \quad . \tag{2.9}$$

Equ. (2.8) also indicates, that the integral vanishes when the radial profile of the azimuthal velocity component is that of a potential vortex, since then the expression $(rw)_r$ vanishes, when the radial distribution of the azimuthal velocity component is given by $1/r$, i. e.

$$(rw)_r = 0 \tag{2.10}$$

If the radial extent of the vortex core is to be described it is necessary to define the diameter D. It is usually assumed that D is given by the location of maximum of the azimuthal velocity w. Outside the core w may decay weaker or stronger than the azimuthal velocity component of the potential vortex for axially accelerated and decelerated flows.

3 Remarks on the numerical solution of the Navier-Stokes equations

The motion of a viscous compressible gas is governed by the conservation equations for mass, momentum, and energy, usually referred to as the Navier-Stokes equa-

tions. In dimensionless form these equations read when transformed on to general curvilinear coordinates ξ, η, and ζ

$$A \cdot \vec{Q}_t + \vec{E}_\xi + \vec{F}_\eta + \vec{G}_\zeta = 0 \tag{3.1}$$

For gases the quantity A is the identity matrix. The vector of the conservative variables Q when multiplied by the Jacobian of the coordinate transformation J is given by

$$\vec{Q} = J(\rho, \rho\vec{u}, \rho e)^T \tag{3.2}$$

In equ. (3.1) ρ denotes the density of the gas, $\vec{u}$ the velocity vector $\vec{u} = (u, v, w)^T$, and e the internal energy. The flux vectors $\vec{E}$, $\vec{F}$, and $\vec{G}$ consist out of two parts, the convective and the viscous part, e. g. : $\vec{E} = \vec{E}_c - \vec{E}_v$, with

$$\vec{E}_c = J(\rho U, \rho U u + \xi_x p, \rho U v + \xi_y p, \rho U w + \xi_z p, U[\rho e + p])^T \tag{3.3}$$

$$\vec{E}_v = (J/Re) \begin{pmatrix} 0 \\ \xi_x \sigma_{xx} + \xi_y \sigma_{xy} + \xi_z \sigma_{xz} \\ \xi_x \sigma_{xy} + \xi_y \sigma_{yy} + \xi_z \sigma_{yz} \\ \xi_x \sigma_{xz} + \xi_y \sigma_{yz} + \xi_z \sigma_{zz} \\ \xi_x E_{vs} + \xi_y E_{vs} + \xi_z E_{vs} \end{pmatrix} \tag{3.4}$$

In equs. (3.1) and (3.4) ξ_x, ξ_y, and ξ_z are the metric terms of the coordinate transformation, U is the contravariant velocity, Re is the Reynolds number and σ the stress tensor $\vec{E}_{vs}$ is the dissipative part of the energy flux containing contributions of the stress tensor and the heat flux.

For incompressible flows with constant viscosity and thermal conductivity the Navier-Stokes equations simplify significantly. The energy equation can then be decoupled from the conservation equations for mass and momentum. If the temperature distribution remains uniform throughout the entire flow process, the energy equation can be omitted. The vector of the conservative variables in equ. (3.2) reduces then to a vector, which has only four components, namely pressure and velocity, i. e.

$$\vec{Q} = J(p, \vec{u})^T \quad . \tag{3.5}$$

The matrix A in equ. (3.1) becomes singular for incompressible flows, which renders the numerical integration of the conservation equations more difficult. For fluids with constant density the vectors of the convective and diffusive fluxes reduce to

$$\vec{E}_c = J(U, Uu + \xi_x p, Uv + \xi_y p, Uw + \xi_z p)^T \tag{3.6}$$

and

$$\vec{E}_v = (J/Re) \begin{pmatrix} 0 \\ g_1 u_\xi + g_2 u_\eta + g_3 u_\zeta \\ g_1 v_\xi + g_2 v_\eta + g_3 v_\zeta \\ g_1 w_\xi + g_2 w_\eta + g_3 w_\zeta \end{pmatrix} \tag{3.7}$$

with

$$g_1 = \xi_x^2 + \xi_y^2 + \xi_z^2, \quad g_2 = \xi_x \eta_x + \xi_y \eta_y + \xi_z \eta_z, \quad g_3 = \xi_x \zeta_x + \xi_y \zeta_y + \xi_z \zeta_z \tag{3.8}$$

To advance the solution in time for compressible flows, an explicit Runge-Kutta multistep method is often used. Alternatively an implicit dual-time stepping procedure can be used for the solution of problems, for which an explicit scheme becomes inefficient.

For the simulation of incompressible flows, Chorins method [9] is often applied. In that method an artificial equation of state is introduced that couples the pressure with an artificial density in order to eliminate the singularity of the matrix A. Then the continuity equation contains an artificial time derivative for the pressure which vanishes for steady-state solutions. In [10] and later in [11] this method was extended to unsteady flows by introducing an artificial time τ and adding a pseudo-time derivative $A \cdot \vec{Q}_\tau$ to equ. (3.1). Thereby the pressure is coupled with the velocity and the conservation equations for incompressible flow can be integrated in a fashion similar to the integration for compressible flow. Since the steady solution is computed for the artificial time t the pseudo-time derivative $A \cdot \vec{Q}_t$ vanishes, and the solution is obtained for the unsteady flow in the physical time t.

The discretization of the conservation equations with the aid of a Taylor-series expansion yields a system of linear algebraic equations of the form

$$LHS \cdot \Delta \vec{Q}^{(\nu)} = RHS \tag{3.9}$$

which has to be solved for each artificial time step. In equ. (3.9) $\Delta Q^{(\nu)}$ is the change of the primitive or conservative variables within one time step, the quantiy LHS contains the discretized spatial derivatives of the Jacobian matrices which result from the linearization, and RHS contains the discretized space derivatives of equ. (3.1) and a second- or even higher-order approximation of the physical time derivative. Details of the linearization are given in [10] and [11]. The resulting algebraic equations may be solved with an alternating Gauß-Seidel line relaxation method.

In order to preserve the conservative properties, equ. (3.1) is discretized for a finite control volume. A corresponding difference operator, e. g. δ_ξ for replacing the derivatives of the fluxes for a volume element in a node-centered scheme at a point (i, j, k) reads:

$$(\delta_\xi \vec{E})_{i,j,k} = (\vec{E}_{i+1/2,j,k} - \vec{E}_{i-1/2,j,k})/(\Delta\xi) \tag{3.10}$$

The formulation of the fluxes at the half points $i \pm 1/2$ determines the properties of the discretization scheme, and different formulations may be used for various reasons. For example a cell-vertex scheme with artificial damping may be used for subsonic compressible flows, an adapted upwind discretization may be used for Large-Eddy-Simulation of turbulent flows. For incompressible flows, the projection of the variables to the cell interfaces can be carried out, for example, with the

QUICK-scheme proposed by Leonard. In the simulation of high-Reynolds number flows, high-frequency oscillations may occur. They can be avoided to a certain extent by fourth-order damping terms added to the continuity equation, when discretized with central differences. The latter are also used for the discretization of the Stokes stresses and the Fourier heat flux for compressible and incompressible flows.

Since most of the flow problems to be solved are large in size, the algorithms sketched out above are parallelized by dividing the entire domain of integration into blocks and by assigning each of the blocks to a processor of a parallel computer. Then the difference between a sequential and a parallelized algorithm consists in handling the data on the block boundaries. The data between neighboring blocks are exchanged by sending and receiving data with a message passing library. The communication mode is asynchronous blocking. For global communication a binary tree topology is used in order to minimize the time for exchange of the data. If the grid blocks differ in size, a perfect load balancing cannot be achieved on parallel machines, if only a single job is processed. On the NEC SX-4, for example, perfect load balancing can easily be achieved, since several jobs can be executed in share mode on the same processor.

The visualization of results is a problem by itself in the sense, that it is a priory not known, what flow quantities should be visualized. Since in the process of visualization a large amount of data has to be handled, in most cases the visualization is also carried out on a parallel computer, as for example the computation of streak lines. In order to minimize the communication time between the blocks, all data to be sent to a neighboring block may be packed into a single array and then communicated. For particle tracing, perfect load balancing is very difficult to achieve, as the number of particles is usually not known at the beginning and may also change rapidly within a block. Nevertheless, machine efficiencies of about 80 percent can be obtained.

In general for solutions of the Navier-Stokes equations, as for example the one developed in [12] for three-dimensional, time-dependent, compressible, viscous flows long computation times are to be expected. The solution is written for contour-fitted, block-structured coordinates with a time-dependent grid and an explicit finite-volume integration of second-order accuracy. Typical computations usually involve more than one million grid points. The storage capacity required is about 2 Gbyte, and typical computing times are 120 h per flow field on a single processor.

4 Examples of flow simulations

Many flows carry vorticity, often enforced on them by solid walls, and as a consequence, vortices, in particular vortex rings can be formed. A steady state of the flow may not be reached, even when there is no change in the boundary conditions in the course of time. Several flow modes may even exist, and when the flow undergoes transition from one mode to another the characteristic length and time scales become important. Especially time-dependent flows are therefore, in general, three-

dimensional in nature, and the introduction of symmetry conditions in the solution of internal flow problems, such as axial or equatorial symmetry, may cause severe deviations from the actual flow behavior, that the prediction is far from reality. Despite their enormous technical importance, internal, and in particular vortical flows seem to have received much less attention in the past than did external flows. One obvious reason is that methods of investigation were not available for detailed flow studies. Numerical tools for the description of complex boundary shapes and for devising suitable grids became the subject of systematic investigations only after sufficient computational speed and storage capacity were available, see for example [12]. During the past twenty years, the performance of electronic computing machines could be enhanced by a two orders of magnitude. It is of equal importance to point out, that in the same time the algorithms could be improved to the extent that another two and a half orders of magnitude resulted in the increase of the performance ratio. Presently used numerical solutions of the Navier-Stokes equations are constructed for the simulation of three-dimensional, time-dependent, incompressible and compressible laminar and turbulent flow. Turbulent flows simulations are performed without and with sub-grid models for the description of the Reynolds stresses. When and how sub-grid models are to be used, depends strongly on the nature of the flow.

A first example will illustrate the formation of vortices in a cylinder of a piston engine. For a long time it was believed, that the gas flow in piston engines is fully turbulent and that the flow is not able to form large vortex structures, since the characteristic flow times are very small. Recent investigations, however, show, that vortex rings are being formed during the suction phase. The vortex structures can be captured and traced in the following way: The velocity vectors obtained from the computation for a specified time are projected on an arbitrary plane, for example for the flow in a cylinder either on a meridional or a cross-sectional plane. Then elements of the streamline projections in the plane chosen are constructed from the projections of the velocity vectors. Vortex cores can readily be identified by searching for closed or spiraling streamline segments. After a core has been identified, vorticity vectors can be computed in the neighborhood of the core by differentiation of the velocity vectors, and elements of vortex lines by numerical integration. The step-size of the integration is determined by the distance to a neighboring plane, for which vorticity vectors are obtained in the way just outlined. This method is described in detail in [13]. Figs. 1 and 2 demonstrate the capturing and tracing of vortex cores in a flow of a piston engine during the intake phase. In Fig. 1 a numerically generated time-dependent computational grid is depicted for a 4-valve cylinder [13].

With the velocity distribution obtained by numerical integration of the Navier-Stokes equations on the grid shown in Fig. 1, the instantaneous location of the cores of the vortices generated in the vicinity of the intake valve can be determined as outlined above. In the left part of the following Fig. 2 the vortex cores are indicated by closed and spiraling streamline segments in a plane parallel to a meridional plane of the cylinder, as determined from the computed velocity distribution. The grey

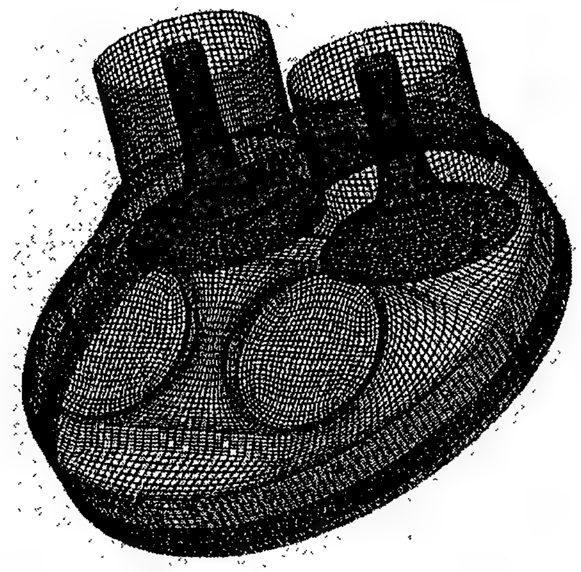

Figure 1: Numerically generated computational time-dependent grid for the simulation of the flow during the intake phase in the piston of a 4-valve engine [13].

straight lines, barely visible, under the first valve on the left of the picture represent exemplary vorticity vectors computed for the innermost streamline segment. The vorticity vectors are then used to obtain an element of a vortex tube. At its end another plane is chosen for which new streamline segments and vorticity vectors are computed, and a second element of the vortex tube can be determined. This procedure can be continued, until the entire core of a vortex is determined. In the picture on the right, the two vortex rings, formed under the open intake valves, are shown. They were computed from the instantaneous vorticity distribution of the time-dependent flow field [13].

The technical importance of the simulations just described need not be emphasized. A better understanding of the unsteady, three-dimensional, compressible flow in piston engines during the intake stroke is crucial for future development and improvement of piston engines with higher performance and lower emission rates. The flow at the end of the compression phase determines the flame propagation speed in homogeneous charge spark-ignition engines, and the fuel-air mixing and burning rates in Diesel engines. The flow has a major impact on the emission values and on the breathing capacity of the engine and therefore also on the maximum available power. The few considerations may suffice to show how important numerical integration techniques deviced for high-performance computing machines will be for future technological developments.

In a second example the question, how accurate the generation and the motion of slender vortex rings and their mutual interactions can be predicted numerically, will be addressed. This question is decisive for technical applications. For this reason the motion of vortex rings is presently being studied by many research groups, not

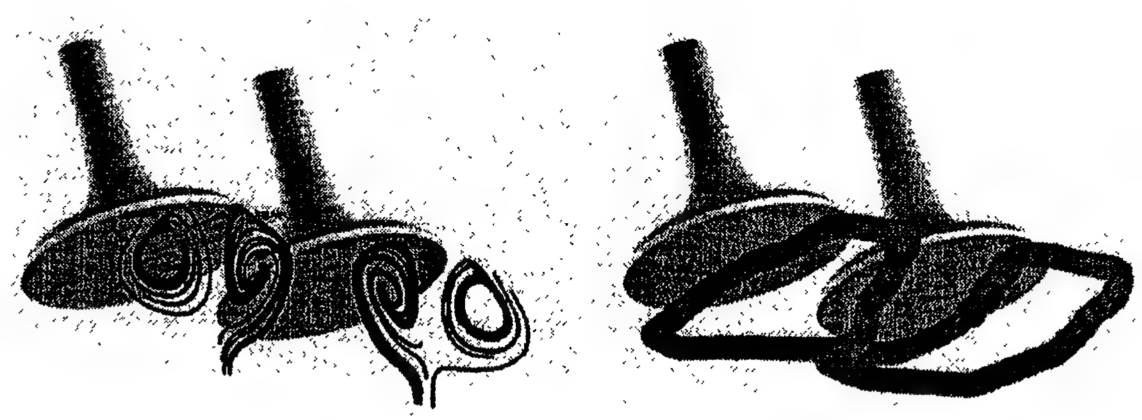

Figure 2: Vortex cores in the intake flow of a piston engine, indicated by closed and spiraling segments of streamline projections on a selected plane in the left part of the picture, with a few vorticity vectors shown near the core by the grey lines on the left. The two ring-like vortices generated by the intake flow are shown in the right picture [13].

only for obtaining a better understanding of vortex dynamics, but also for validating numerical predictions with available experimental data. One such set of experimental data was provided by T. T. Lim in [14]. In his experiments he let two vortex rings of equal strength collide with each other, starting under a certain prescribed angle. The two rings were dyed with different colors, so that the changes taking place during the collision process could be detected. Photographic studies revealed, that the two rings first merge into a single ring, half of it showing the color of one of the starting rings, and the other half that of the other. After some time, the newly generated vortex ring is stretched in the direction normal to direction of the original motion, and two new vortex rings are formed. They separate from the intermediate one-ring structure, again in the direction normal to the direction of the initial motion of the two starting vortices.

This problem, obviously complex in the details of the collision process, was studied with a numerical solution of the Navier-Stokes equations by J. Hofhaus at the Aerodynamisches Institut in order to investigate the accuracy of the numerical predictions by comparing them with the experimental data of [14]. The results are reported in [15]. Several difficulties had to be overcome in order to make the comparison possible. For example, because of the motion of the vortex structures in the flow field, their instantaneous location must be controlled. In particular, it must be avoided, that they touch the boundary of the domain of integration during the numerical simulation, as then the solution can no longer converge. In [15] this difficulty was circumnavigated by introducing a moving system of coordinates. Since the velocity of the vortices can be determined, they can always be kept in the center of the domain of integration, at least at a safe distance away from the boundaries.

Other problems arise from the necessity to prescribe initial and boundary conditions. It is certainly possible to follow the experiment and release a certain volume of a

incompressible fluid through an orifice. Such an approach would, however, require a large number of grid points for the description of the flow during the initial phase. For later phases of the time-dependent flow these points can no longer be used. Another disadvantage of this approach might result from grid singularities, associated with circular cross-sections with corresponding coordinates.

Before comparing some of the numerically predicted results with experimental visualizations, the mechanism of the collision of two vortex rings is schematically explained in Fig. 3, taken from [15]: Two vortex rings of equal strength approach

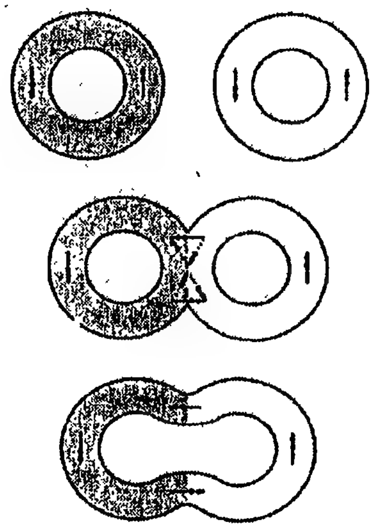

Figure 3: Schematic of collision of two vortex rings. Opposite signs of vorticity vectors, indicated by the small arrows, retard the rotation locally in the plane of contact and cause the merging into a single vortex structure [15].

each other with the same speed. The components of the vorticity vectors parallel to the plane of contact have opposite signs and retard the rotation of the flow particles during the collision, and the two vortex rings form a single one.

If the strength of the vortices is large enough, the distant parts of the newly generated structure, rotating in opposite directions, begin to interact, and two new vortex rings may be generated, separating in the direction normal to the original direction of motion. The shape of the structure that is formed in the collision depends to a large extent on the angle under which the two vortex rings are initially set in motion. This can be seen in Fig. 4, where results of numerical simulations are shown for a Reynolds number Re $= 500$ and initial angles between the axes of the vortex rings, j. The Reynolds number is based on the initial radius of the vortex ring and its initial speed. Plotted are the vortex lines for angles $\phi = 40°, 50°, 60°, 70°, 80°$, and $90°$. An analysis of the numerical data showed, that in the process of redistribution of vorticity only a certain part of the strength of the original vortex rings is found in the newly formed vortex pair. As already indicated through the connecting vortex lines in Fig. 4 for an initial angle $\phi = 80°$, a third vortex structure with closed

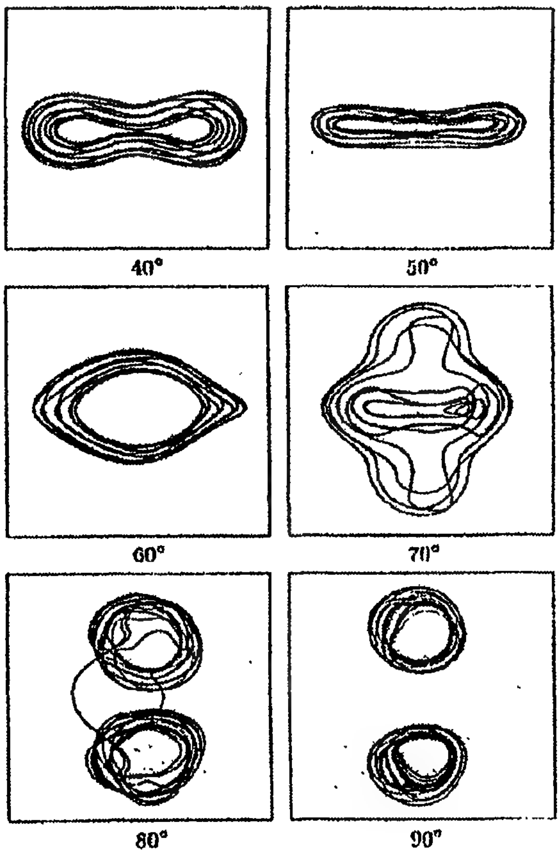

Figure 4: Formation of vortex structures after collision of two initial rings at a Reynolds number Re = 500 and various initial angles of the collision [16].

vortex lines is formed on the leeward side of the vortex pair. The comparison with the experimental data of [14] in Fig. 5 shows, that this redistribution of vorticity is also observed in the actual flow.

The third structure results from the interaction of the vortex pair generated in the collision, as the two vortex rings (dark structures in Fig. 5) do not lie in one and the same plane. They can therefore continue their spatial redistribution of vorticity through interaction. The comparison in Fig. 5 also confirms, that the generation of vortex rings and their mutual interaction can be predicted with numerical solutions of the Navier-Stokes equations, provided the Reynolds number is low enough so that the details of the interaction process can be resolved with grids that can presently be implemented on available high-performance computers.

5 Concluding Remarks

The occurrence of slender vortices in various areas of fluid dynamics was demonstrated, in particular for aerodynamics but also for other technically important flows. Results of investigations of incompressible and compressible flows contain-

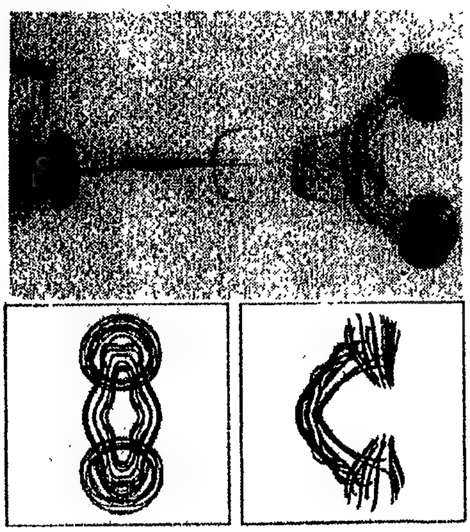

Figure 5: Comparison of the computed results of [15] (lower picture) with the experimental data, of [14] (upper picture). A third vortex structure with closed vortex lines is formed on the leeward side of the vortex pair, (dark spots) as a consequence of their interaction.

ing structures of slender vortices were presented. The examples chosen comprise the formation and interaction of vortex rings in cylinders of automotive engines during the suction phase and the oblique collision of vortex rings.

It was shown that for the cases considered numerical simulations of time-dependent three-dimensional vortical flows are in good agreement with experimental data, for external flows as well as for internal flows, as long as sufficient numerical resolution of the local flow phenomena can be provided. The results were obtained with solutions of the complete Navier-Stokes equations. Description of the algorithmic elements of the solutions employed was purposely kept short in order to be able to emphasize the comparison of the results with experimental data.

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Numerical results for the CGBI method to viscous channel flow

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Abstract

The *conjugate gradient boundary iteration* (CGBI) is a domain decomposition method for elliptic partial differential equations [1] [2] [3] [6] [7]. In the context of fractional step methods [10], a Navier-Stokes solver based on CGBI and a characteristic's method was constructed. In this article, numerical results for the incompressible flow past a backward facing step and past a cylinder (both modelled in 2d) are presented.

1 Introduction

For the efficient computation of the flow past an obstacle in a channel, the possibility to combine the high accuracy of spectral solvers with the high flexibility of the finite element method (FE,FEM) seems desirable. The CGBI parallelization method is a tool to couple such heterogeneous solvers for elliptic problems. The channel is devided into several non-overlapping subdomains, where Chebychev collocation spectral solvers [4] are applied on rectangular subdomains and FE solvers on the subdomains containing the obstacle (Fig. 1). This article shows the application of CGBI for the computation of a Navier-Stokes flow. The time-splitting scheme decouples each Navier-Stokes timestep into a transport step and elliptic problems for the velocity and for the pressure. CGBI is used to solve the elliptic problems, and the transport equation is handled by the characteristic's method.

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Γ_1 Γ_2 Γ_3 Γ_4 Γ_5 Γ_6

obstacle

Ω_1 Ω_2 Ω_3 Ω_4 Ω_5 Ω_6 Ω_7

Figure 1: Domain decomposition of a channel with an obstacle inside.

2 The CGBI-characteristics solver

The time scheme. Concerning the time scheme, a pressure correction method [2] [6] [7] [10] is used to discretize the Navier-Stokes equations. Each timestep is splitted into three parts: The transport step, which is solved by the characteristic's method, and two elliptic problems, i.e. the diffusion step and the pressure correction step, which both are solved by the CGBI method.
The diffusion step leads to resolvent type equations

$$(\sigma - \Delta)w = f, \quad \sigma \geq 0, \tag{2.1}$$

for the velocity components with Dirichlet boundary conditions on the physical boundary, whereas the pressure equation leads to a Poisson equation (i.e. $\sigma = 0$ in (2.1)) with Neumann boundary conditions.
As the Lagrangian ('material') derivative is presently approximated by a first order difference quotient, the Navier-Stokes time scheme is presently only of first order in Δt; the implementation of a higher order scheme is straightforward and carried out in the near future. Presently, we concentrate on the spatial accuracy.

The CGBI method. CGBI is a domain decomposition method for the parallel solution of the resolvent equation (2.1). The theory of the CGBI method together with numerical results for elliptic problems was subject of former publications [2] [3]. Let us summarize the main features:
The occuring local problems on the subdomains $\Omega_1,\dots,\Omega_p$ use boundary conditions of Neumann type on the interfaces $\Gamma_i = \bar{\Omega}_i \cap \bar{\Omega}_{i+1}$. The correct Neumann boundary data on the interfaces $\Gamma := \bigcup_{i=1}^{p-1} \Gamma_i$ happen to be the solution of a certain minimization principle. This minimization principle is solved by a conjugate gradient method (CG) running on the interfaces Γ. In each CG step, local problems on the subdomains being *independent* of each other are to be solved.
Efficient preconditioners using only $O(N \log N)$ operations for $N \times N$ grid points per subdomain were developed for CGBI. Basically, these preconditioners are approxi-

mations of the square root of the negative Laplacian operator on the interfaces [6] [7]; they do *not* require the solution of any subdomain-based problem. These preconditioners lead to a condition number *independent* of the number of subdomains and of the number of unknowns and of σ. If the subdomains are not too narrow, an error reduction rate of 0.1 per CG iteration step was found, both for the resolvent equation and for the Poisson equation. That means that ~ 8 CG steps are to perform if spectral solvers are used within CGBI, and ~ 4 if only lower order solvers are used such as FEM, FDM.
The CGBI methods resembles the *dual Schur method* (e.g. [5]); however, the preconditioning and the treatment of the 'floating' subdomains (i.e. subdomains with pure Neumann b.c. for the Laplacian equation) is different.

The characteristics solver. Our characteristics solver traces the characteristic line for each grid point backward in time by the classical 4^{th} order Runge-Kutta scheme. For the tracing and the computation of the characteristic foot point, interpolation in space and time is required. Our algorithm uses linear interpolation in time. For the spatial interpolation on the spectral domains, we discard the full order spectral interpolation because of its large computational costs. Instead, we use piecewise polynomials of order $p = 1, 2, 3$. On FE domains, the piecewise linear FE representation is used.
Numerical tests revealed no stability restriction of the timestep size. At least for equidistant and quasi-uniform meshes, this could be expected due to theoretical stability investigations.

3 Numerical results

Flow over a backward facing step. As a first example, we present the computation of a flow past a backward facing step. The Reynolds number is moderate so that the flow becomes stationary after some time. The reason to choose this example is the following: As we focus on the stationary state of the flow, the time discretization error of our scheme is of minor importance; we can even choose the rather large time step size $\Delta t = 0.08$. This test case enables us to investigate the spatial accuracy of the Navier-Stokes solver.
In a first stage, our domain is rectangular of size

$$\Omega = (0,6) \times (0,1), \tag{3.1}$$

and the step of height 0.5 is modelled by posing a Poiseuille inflow profile on *the half* of the left edge of the computational domain. This simplified geometry of the computational domain enables us to use the spectral solver on *all* subdomains. The maximum inflow velocity is 1.0, and the Reynolds number with respect to the step size and the maximum velocity at the inlet is $Re = 150$. We compare the test runs of

1. Chebychev solvers on all 6 subdomains ('CC'-coupling), interpolation in the transport step by polynomials of order $p = 1$

2. Chebychev solvers on all 6 subdomains ('CC'-coupling), interpolation in the transport step by polynomials of order $p = 3$
3. FE solver on first subdomain, Chebychev solvers on the other subdomains ('FC'-coupling), $p = 3$ on the Chebychev domains,
4. FD solvers on all subdomains ('DD'-coupling).

In a second stage, we include the upstream part of the channel into the computational domain in order to make a computation comparable to the benchmark [8]; our new computational domain Ω is L-shaped:

$$\Omega = (0, 0.75) \times (0.5, 1) \cup (0.75, 6) \times (0, 1). \tag{3.2}$$

It is devided into 6 subdomains of the same width; on the first, the FE solver is used. This geometry and the Reynolds number correspond to the benchmark tests in [8]; except that the length of our channel is smaller. However, our computational results showed that the influence of a longer channel in downstream direction on the numerical result is neglectable. The spectral solvers use $(N+1) \times (N+1)$ grid points on each subdomain, $N = 8, 16, 32, 64, 128$, and on the FE domain, a regular mesh of triangles with a similar number of nodes is used. See Fig. 3 for the visualization of the x-component of the flow for both geometries ($N = 64$).
For the analysis of the results, we focus on the length L of the recirculation zone at time $T = 50$. At this time, the recirculation length is already stationary up to ± 0.005. Fig. 2 shows L in dependence of the method and on the spatial discretization parameter N. Fig. 2 reads that all test runs 1.-4. using the rectangular domain (3.1) lead, for $N \to \infty$, to a limit of about 2.67. Furthermore, we see that for the case of Chebychev solvers on all subdomains, the order of the spatial interpolation in the transport step is essential for the accuracy of the whole Navier-Stokes solver: For the high order interpolation $p = 3$, the result for $N = 8$ is more accurate than for the lower order interpolation $p = 1$ and $N = 64$. Fig. 2 also shows that the influence of the lower order FE domain onto the error of L is limited; the 'FC' result is clearly more accurate than the 'DD' result.
Let u_N denote the numerical solution for $(N+1) \times (N+1)$ grid points per subdomain. Assuming a law

$$||u_N - u_{exact}||_{L^2(\Omega)} \approx c N^{-\alpha} + f(\Delta t) \tag{3.3}$$

for the error, we can compute the order α approximately without knowing u_{exact} by

$$\alpha \approx \log_2 \frac{||u_N - u_{N/2}||_{L^2(\Omega)}}{||u_{2N} - u_N||_{L^2(\Omega)}}.$$

We get the following results:
Let us explain these results for α. The spatial interpolation error in the computation of the characteristic's foot points can be estimated by

$$c \min\{h_{loc}^{p+1}, |u| \Delta t \, h_{loc}^{p}\}$$

which becomes the largest in the center of the subdomains where the Chebychev-Gauss-Lobatto mesh is the coarsest. Here, the Courant number $|u| \, \Delta t / h_{loc} \gtrsim 1$, So

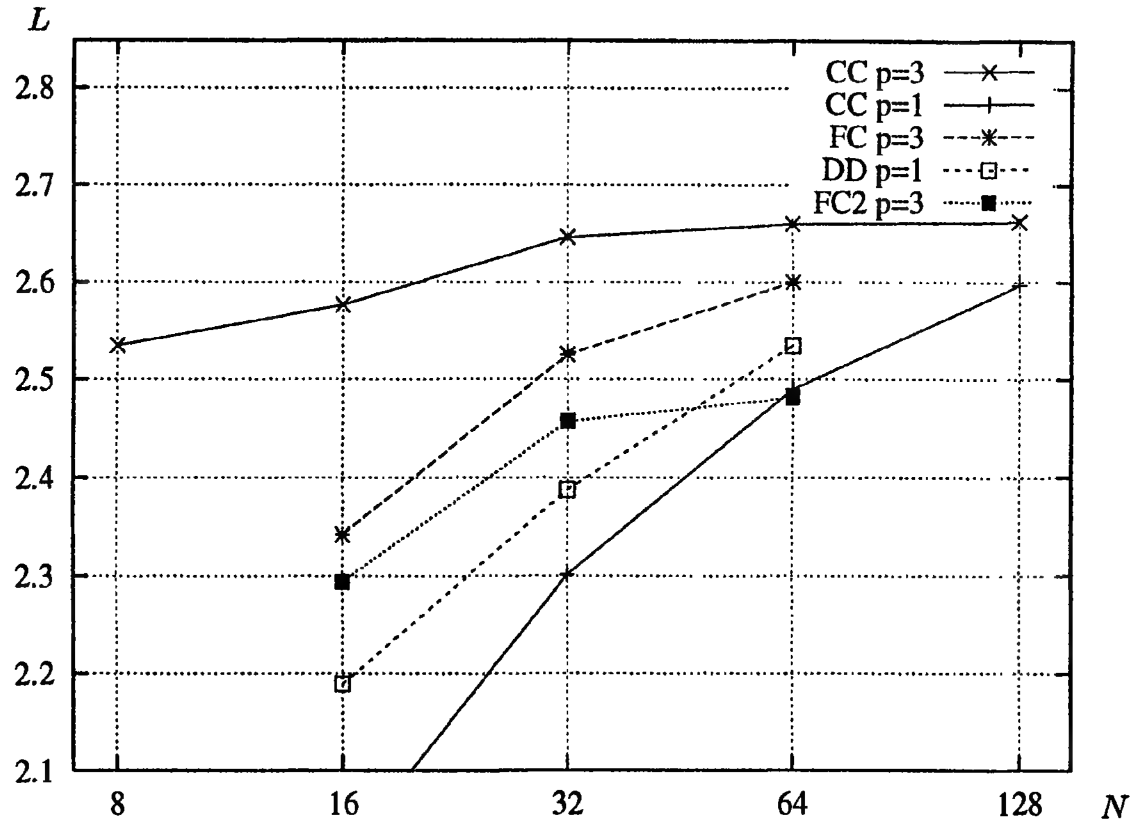

Figure 2: The length of the recirculation zone for different combinations of local solvers. CC=Chebychev method on all subdomains, FC=finite element method on first subdomain and Chebychev on the other subdomains, DD=finite difference method on all subdomains, FC2: as FE, but with the different geometry (3.2). p = order of polynomial for spatial interpolation on Chebychev domains for the characteristic's method.

we can expect the order $\alpha = p$ with respect to $h_{loc} \sim N^{-1}$. However, we cannot expect that the Δt-dependent terms of (3.3) extinct each other completely, in which case a smaller value for the order α is pretended.
Results very similar to Table 1 are gained by regarding $L(N)$ instead of the $L^2(\Omega)$-error of u_N.
Finally, let us compare our test runs 'FC2' on the L-shaped computational domain (3.2) with the benchmark [8]. We find a good correspondence of our results; in [8], most results are situated between 2.2 and 2.6; using a discretization comparable to [8] ($N=32$), 'FC2' in Fig. 2 reads 2.46.

Channel flow past a cylinder. In the following we present the computation of a non-stationary 2d benchmark problem introduced in [9]. A circular obstacle of diameter 0.1 is situated in a channel of width 0.41. The center of the obstacle is positioned 0.20 from one channel border and 0.20 from the entrance of the channel. At the inflow, a Poiseuille profile $u(y) = 6\,U_{mean}\,y\,(1-y)$, $U_{mean} = 1.0$, for the x-component of the velocity is prescribed. The Reynolds number $Re = U_{mean}D/\nu$

method	order α
CC (p=3)	2.6
CC (p=1)	0.8
DD	0.7

Table 1: Approximate order of the different methods with respect to the length of the recirculation zone for the backward facing step

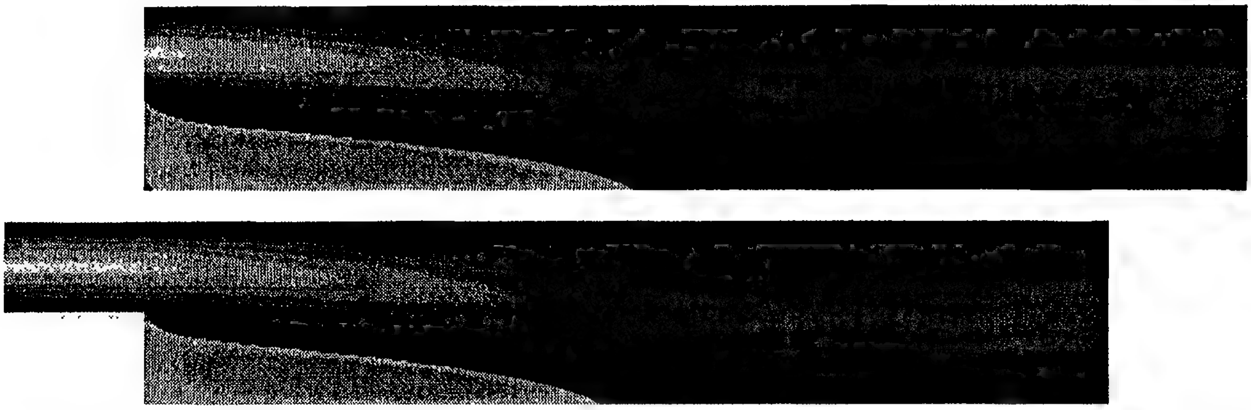

Figure 3: x-component u of the velocity for the flow over a backward facing step for the two different computational domains (3.1) and (3.2). For positive values of u, the brightness indicates the quantity. Negative values of u (the recirculation zone) are coded with bright colors again. Upper part: the computation on 6 spectral subdomains ('CC'), lower part: the computation on 1 FE and 5 spectral subdomains ('FC2').

with respect to the mean velocity at the inlet and the diameter D of the obstacle is $Re=100$.
Our computation uses 6 square subdomains. The first, containing the obstacle, uses the FE solver and the other the spectral solver. On the spectral domains, the interpolation by polynomials of order $p=3$ is used during the transport step.
Due to the asymmetric position of the obstacle the periodic vortex shedding establishes rapidly. In Fig. 4, the norm of the velocity, the y-component of the velocity and the vorticity for $N=64$, $\Delta t=0.002$ at time $T=5.0$ are displayed.
Although a coarse spatial discretization on the FE domain (no local refinement close to the obstacle) and the first order time scheme was used, we found a Strouhal number $St = fD/U_{mean}$ (f=frequency of vortex shedding) of $St=0.26$; the benchmark computation [9] gives $St=0.300 \pm 0.005$ to be the 'exact' value.
We observed an increase of the Strouhal number when a finer temporal and/or spatial discretization on the FE domain was used. See the next chapter for current improvements.

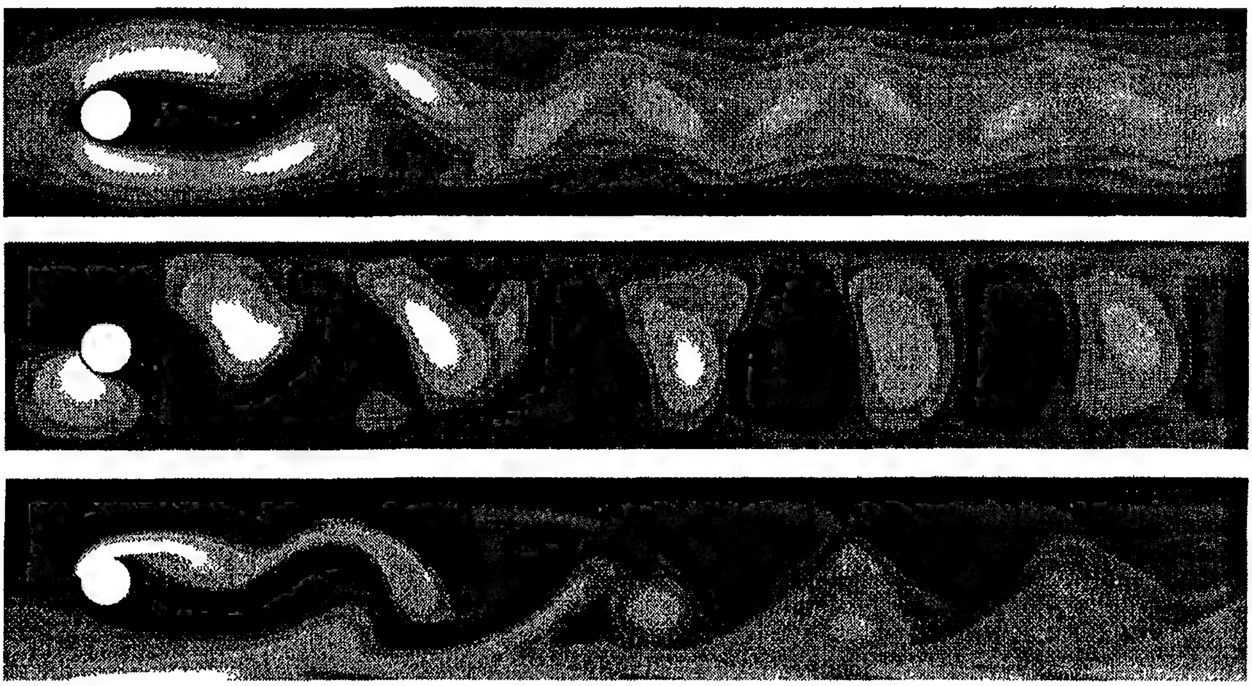

Figure 4: The norm of the velocity, the y-component of the velocity and the vorticity for $N=64$, $\Delta t=0.002$, $p=3$, $T=5.0$.

4 Summary and Outlook

We think that CGBI is a well suited parallelization tool for the coupling of FE solvers and spectral collocation solvers, and the highly accurate computation of a Navier-Stokes flow. A high spatial accuracy is reached on the spectral domains. Presently, the FE solver uses approximately the same amount of grid points as the spectral solver. To balance the error and also the processor load, we intend to use a refined FE mesh (which requires a preconditioning or a multigrid method for the local FE problem) for future computations.
The implementation of a higher order time scheme is presently under investigation. Without much modification of the code, a higher order characteristic's method (i.e. a higher order of the approximation of the Lagrangian derivative) is possible. Another possibility which is under current investigation is the introduction of a semi-Lagrangian method [11] which avoids costly spatial interpolations. However, the semi-Lagrangian method leads to a restriction of the timestep size used in the transport equation, but we intend to overcome this problem by 'subcycling' [11].

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Isothermal Drop-Wall Interactions. Introduction to experimental and numerical studies

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1 Introduction

The impact of drops onto a dry or wetted surface can have great importance in technical applications and natural phenomena such as soil erosion, turbine blades corrosion, surface cooling, formation of charged droplets in a thunderstorm, dispersal of spores and micro-organisms, spray injection in diesel and gasoline direct-injection engines, liquid painting. The impact of liquid droplets onto a solid surface, either dry or wetted, has been studied for more than one century. However, because of the high complexity of the splashing and spreading processes, a detailed and satisfactory description of these phenomena is still not available in the scientific literature. A drop impinging on a surface can spread over it, rebound or form a crown (Fig.1a-e). Several other phenomena may also occur. For example, as shown in Fig. 1b-c, during crown formation, some jets can be produced and then break, thus throwing away secondary droplets. In Fig. 1d a lamella breaks up during the receding contraction of a drop on a non-wettable surface. Finally, if the surface is covered by a liquid film or has a low wettability, a conical rebounding jet is formed which in turn can produce a few secondary drops as an effect of the elastic collision (Fig. 1e-f). The dynamics of the impact of a single liquid drop on a wall depends on the physical parameters of the impinging droplet, on the solid surface properties and on the near wall boundary conditions. In particular, the impact process is influenced by the drop dynamics (its diameter D, velocity V normal to the impacted surface, impact angle α), liquid properties (non-Newtonian or viscoelastic fluid, viscosity μ, density ρ, surface tension σ) and droplet internal conditions (circulation, presence of solid particles, non-homogeneous density or viscosity). The main surface parameters are its temperature, wettability, roughness, porosity, elasticity, charge, curvature and, if the surface is wetted, the film thickness, its waviness and chemical composition. The near-wall conditions can influence the impact dynamics mainly through the presence of an air boundary layer or a vapor film. Moreover, the im-

pact frequency has to be considered as an independent parameter, since its influence is not only the mere production of a liquid film of a given thickness on the solid surface. A time scale should be included in the dimensional analysis: its standard adimensional form is the "convective" time scale $\tau = Vt/D$.

Consider a droplet of a Newtonian homogeneous liquid, then the evolution of the flow of the liquid film after the impact is the result of the opposing action of surface and inertial forces, damped by viscous stresses. In some circumstances, gravity may also play a role [Cossali et al. (1997)]. Adimensional quantities that play a major role in the droplet dynamics are the Weber number ($We = \rho V^2 D/\sigma$), the Ohnesorge number ($Oh = \mu/\sqrt{\rho D\sigma}$), the Reynolds number ($Re = \rho VD/\mu$) and the Froude number ($Fr = V^2/Dg$). Only three of these numbers are independent. The velocity entering the previous definitions is the component normal to the wall or its module, but independently from this choice the impact angle is important [Faddeyev et al. (1988), Marengo et al. (1998), Stow and Hadfield (1981)] and its influence must be considered separately. Another relevant parameter is the drop deformation: the variable D is generally written via the square root of the maximum area cut by the droplet on a plane normal to the direction of motion. Finally, the drop dimpling just before the impact should be checked and considered in the analysis.

Among surface properties and "interaction" features, we mention a few adimensional parameters: surface roughness ($R_{nd} = R_a/D$), film thickness ($\delta = h/D$) and, for a monodisperse drop-chain impact, impact frequency ($f_{nd} = fD/V$). The Bond number ($Bo = \rho g h^2/\sigma$) may also have some relevance. A few authors investigated other features like surface curvature, charge, elasticity and a non-homogeneous liquid in the impacted film. The following discussion will provide a review of studies about single droplet impact on a cold solid surface and on a homogenous liquid film.

2 Spreading and Splash Evolution: Experimental Classification

For impacts on wetted surfaces it is possible to distinguish four main temporal phases: (1) the lamella formation and the prompt splash, (2) the crown rise, the cavity and the vortex rings formation, (3) the jet formation and break-up (*late splash, secondary droplet formation*) and (4) the crown collapse and a sort of elastic rebound. For droplets impacting on a solid surface, there are three spreading stages: the *kinematic phase* $\tau < 0.1$, the spreading phase and a final phase during which the lamella decelerates strongly and reaches an asymptotic diameter. A clear separation between impacts onto dry and wetted surfaces is necessary since the morphology of spreading and splashing are rather different in the two cases.

2.1 Drop Impacts on Wetted Surfaces

Drop deformation, lamella ejection, prompt splash

Drop deformation, lamella ejection, prompt splash The lamella formation (a radial flow departing from the impact point) was first detected by Worthington (1876,1877a,1877b,1897). With an innovative photographic technique, he studied the formation of the crown and jets, with evidence of perturbations along the crown and break-up of the jets. The protruding of a liquid lamella from the deformed drop surface is still an unsolved problem. The influence of liquid compressibility was considered by several authors [Engel (1967), Heymann (1969), Field et al. (1985)].

The liquid film on the surface deforms slightly when the falling drop approaches, then the droplet preserves its spherical form for a longer time than in the case of an impact on dry surfaces. Hence with a good approximation the drop apex interacts with the surface at $\tau = 1$. The lamella speed, when it can be determined from experiments, is about 6-10 times larger than the impact velocity V and shows a

decreasing power-law behavior with time. The lamella can produce a crown or a simple radial flow. The advancing rim is moving with a vertical velocity component which depends on the film thickness [Marengo (1996)]: the thicker the film, the higher the angle between the lamella and the surface. The rim can show waviness and some fingers at this early stage as well. The lamella evolves toward a vertical cylindrical shape or crown shape. During this evolution, some droplets can also be produced by early hydrodynamic instabilities (Fig. 3). This effect was noticed in splashes of water drops [Cossali et al. (1997)] and called "prompt splash". At lower surface tensions, the prompt splash appears to be amplified [Rioboo et al. (1999)].

Crown evolution

The crown formation has been theoretically analyzed in the case of impact on wetted solid surfaces. Energy balances have been used to characterize the crown evolution and to evaluate the importance of different physical parameters. However, dissipation is very difficult to quantify and the comparison between experimental data and theoretical prediction is poor. More recently two models, based on force balances, have been proposed with some success [Yarin and Weiss (1995), Roisman et al. (2000)]. Yarin and Weiss study impact over a thin liquid layer. They describe the evolution of a horizontal radial flow using a thin layer approximation. Their analysis is equivalent to the potential flow analysis for jet impacts performed in a similar context by Peregrine (1981). It yields for the crown diameter the estimate

$$\frac{r_c}{D} = Re^{1/4}(\tau - \tau_0)^{1/2} \tag{2.1}$$

where τ_0 is a shifting factor that takes into account the experimental error to predict the impact time. However, Peregrine (1981) noticed that: a) potential flow theory for jet impacts yields lamellas that are not vertical but inclined at an angle with the surface (this is at variance from observations both numerical and experimental); b) potential flow theory does not necessarily apply. If on the other hand potential flow theory is abandoned, then relation [2.1] is not the only solution to the jet impact. This may explain why a lower exponent for the crown radius evolution after impacts of water droplets was found experimentally. Indeed, Coghe et al. (1999) found $r_c/D = C(We)(\tau - \tau_0)^{0.4}$, for $\tau \leq 15$, where C $\approx$ 0.7 and slightly increases with Weber number.

The analysis of the crown evolution generated by water droplets impacting on a thin liquid film showed that the adimensional vertical velocities of the crown varies slightly, depending only on the film thickness. On the other hand, the adimensional crown thickness increases in time to 3-5 times the initial value, independently of film thickness and impact velocity. The maximum crown height increases with impact velocity and it is reached in the interval $7 < \tau < 15$ for $2 < V < 4 m/s$ and D = 3 mm. Figure 4 reports the adimensional crown height $\eta_c = H/D_o$ versus adimensional time for some experimental conditions. The crown height reaches a maximum value ($\eta_{c,max}$) after a time τ_{max} from the impact. These two values depend on the drop impact velocity, whereas the influence of the film thickness appears to be very weak. The ratio $\eta_{c,max}/\tau_{max}$ is independent of the drop impact velocity as both $\eta_{c,max}$ and τ_{max} seem to scale similarly with impact velocity: $\eta_{c,max} = A_1 We^n, \tau_{max} = A_2 We^n$ with $n = 0.65 \div 0.75$.

Macklin and Metaxas (1976) used different fluids (methanol, water and glycerol) to correlate the crown geometrical aspect to the Weber num-ber for impacts on both shallow and deep liquid films. The crown maximum height depends on Weber number and varies between 0.5D and 4D, and in general the bigger the Weber number, the thinner the crown and the greater the liquid volume present in the crown. When a drop falls through a liquid surface, usually a cavity forms under the free surface. The papers by Engel (1955,1966,1967) and Macklin (1969,1976) give

some information on this problem for drop impacts with $\delta >> 1$. In principle, an interaction between the solid surface and the cavity bottom cannot be excluded and therefore the solid surface could play a role also for impact on a liquid film [Hobbs and Osherof (1967), Marengo (1996)]. At high impact velocities, liquid jets can develop from the rim on the top of the crown and eventually break. The number of jets protruding from the crown rim decreases with time, while it increases by lowering the liquid surface tension. Once formed, the jets may have a ramification and subdivide in two or three branches. Many theories have been developed over time for jet formation, however we find none of the published theories satisfactory.

The critical splash regimes

The lamella spreading can result in three main phenomena: a recoil with jet formation or drop rebound, a splash with secondary droplet formation, and a simple deposition of the drop on the surface. The determination of empirical relations among impact variables to predict the outcome of the lamella spreading is of outstanding importance in industrial applications and for numerical simulations.

The secondary droplet formation is due to a Rayleigh break-up of the jets. Several authors [Stow and Stainer (1977), Stow and Hadfield (1981), Walzel (1980), Mundo et al. (1995), Cossali et al. (1997), Yarin and Weiss (1995)] indicate the dimensionless group $K = WeOh^{-0.4}$ as the discriminating variable for splashes on both dry and wetted surfaces. When K is greater than a critical value K_{cr}, related to surface conditions such as its roughness, then the splash happens. The dimensionless film thickness plays an important role in splashing. Cossali et al. (1997) determined the critical value of $K_{cr,}$ for thin liquid films $\delta < 1$. Four main regimes can be distinguished: thin film, liquid layer, shallow and deep pool [Tropea and Marengo (1998)]. In Table 1 a survey of the critical K values is given.

Some authors considered also the limit for the conical jet break-up after the crown collapse. Rodriguez and Mesler (1985) found the critical value of K = 790 for the secondary droplet formation from the conical final jet.

The splash products

The outcome of the splash is influenced by impact conditions: (a) the number of secondary droplets increases with the Weber number and decreases with both the Ohnesorge number and the dimensionless film thickness, (b) the mean diameter of the secondary droplets decreases with higher impact velocity, bigger surface roughness and thicker films. It has also been found that the ejection angle of secondary droplets is a function of impact parameters.

Many empirical models define a mean droplet diameter that actually varies in time during the break-up process. The diameter increases in time with the law

$$d_{sec}/D = 4.6\tau^{*1/2} \tag{2.2}$$

where τ^* is defined as

$$\tau^* = V\frac{t}{d}We^{-1}Oh^{1/6}(1+\delta^{11/7}) \tag{2.3}$$

The first secondary droplets are generated at a time τ^* in the range [0.004÷0.008] after the impact.

After reaching its maximal height, the crown starts to collapse. For impacts on a liquid pool and with a Weber number greater than 400, the crown closes off before collapsing. Near the end of the collapse a conical jet forms. Some secondary droplets may still be produced, at least for the most energetic impacts. The height of the conical jet and the number of secondary droplets that it ejects, increase with a decreasing liquid film thickness.

2.2 Drop Impacts on Dry Surfaces

Drop deformation

As for the wetted case, at the early stage of the impact the shape of the droplet is similar to a truncated sphere and no lamella is yet present. After the detachment of the shock wave, a cavitation bubble can appear in the center of the drop [Engel (1966), Chandra and Avedisian (1991)]. Engel was the first to determine the maximum pressure and velocity that this phenomenon can produce. She found that this pressure was proportional to the waterhammer pressure and that this velocity could be up to 8.6 times bigger than the impact one. Moreover, the velocity of the contact line along the wall can be much higher. This problem is related to the known paradox of the edge velocity of a knife cutting a plane. The propagation velocity of the contact line along the surface is clearly infinite at time t = 0. Through a simple geometrical analysis it scales as $V \approx At^{-1/2}$. A recent measure of this velocity is presented in Fig. 5, where a maximum value of about 40 m/s is obtained at $10\mu s$ after the impact. Its ratio with the impact velocity is about 30.

Experimental data show that the spreading diameter grows in time with a power law with an exponent that lies in the range [0.45÷0.57]. Its precise value depends on the liquid and impact conditions, which apparently do not affect the overall shape of the droplet.

Lamella ejection

As time goes on, a lamella is ejected from the base of the droplet and forms a thin film. The lamella is characterized by radial perturbations and for high impact velocities azimuthal undulations appear along its contour. No satisfactory correlation has been found to date to characterize the number of these perturbations as a function of impact and surface parameters [Marmanis and Thoroddsen (1996)]. Lamellae expand and can form a thin liquid crown or spread over the surface till a maximum radius is reached.

In case of high viscous liquids, the droplet is smoothly and continuously deformed and a distinct lamella is no longer visible. By increasing the impact velocity or the drop diameter the spreading is faster, while it is slower by increasing the surface tension or viscosity. For $\tau < 0.1$, drop diameter and impact velocity are the governing parameters. Surface tension plays no role for $\tau < 0.5$, then till $\tau < 2$ the spreading diameter slightly increases with smaller surface tensions. In Fig. 6, we show the time evolution of the spreading diameter for different viscosities ($\tau \approx 1$ and D is kept constant). At higher liquid viscosity the spreading diameter is smaller and its maximum is reached at an earlier time.

A further increase of the impact velocity leads to a splash. Engel (1967) noticed that splashing occurs for less energetic impacts, but with rough surfaces, while Hartley and Brunskill (1958) found that a necessary condition to obtain a droplet rebound was the presence of 'micro-roughness' on the leaves. Range (1995) investigated the influence of surface wettability.

Spreading phase

Wettability has no influence during the initial *kinematics phase*, but as the time increases its effects become important. This is shown in Fig. 7, where only at later times a difference is discernable for the water-repellent case ($\Theta_{rec} = 154^0$). The droplet comes to a rest earlier and with a smaller maximum diameter. Fukai et al. (1995) found that for more wettable surfaces the maximum spreading increases. In case of completely wettable systems (static contact angle= 0^0) and of stationary conditions, the dynamical contact angle is only a function of the capillary number Ca ($= \frac{\mu V}{\sigma}$). For partially wettable systems (static contact angle $\neq 0^0$) the value of the dynamical contact angle is a function of the capillary number and of the static

contact angle [Blake (1993), Kistler (1993)]. At high impact velocities, wettability has a smaller influence on the deposition rate than surface roughness [Podvysotskii and Shraiber (1993)].

Roughness increases the energy dissipation during the spreading, but it promotes secondary atomization processes (splashes). The greater the adimensional surface roughness, the lower the value of K_{cr} necessary for splashing. Other two main additional effects of surface roughness are the higher amplitude and number of perturbations during the spreading and the entrapment of air or vapor between the liquid interface and the solid surface.

Perturbations on the rim appear on rough surfaces even at relatively low impact velocities. By increasing this velocity, perturbations develop on the rim even on smooth surfaces. On a rough glass, a prompt splash occurs at high impact velocity ($V > 3m/s$). The comparison of three images of impacts of water droplets on a PVC plate with different roughness wavelengths (the wavelength is a measure of the mean distance between surface profile peaks. For this experiment the surfaces have been prepared by laser ablation to obtain a "deterministic" roughness) shows that shorter wavelengths lead to prompt splash at lower impact velocities. The limiting velocities for these operating conditions (a water droplet of 3.4 mm) are $V_{imp} = 3.1$ m/s for the surface with roughness $R_w = 300\mu m$ and 2.7 m/s for the surface with $R_w = 100\mu m$.

The final spreading phase

Recently, Rioboo et al. (1999) showed that given a fixed surface, glass in his case, the splash occurs at different K numbers for different fluids, in particular a lower critical value of K for ethanol than for water. As a result, the K number alone is not able to define the splashing threshold. Deposition occurs on a rough glass with water droplet for K equal to 5342 and an adimensional roughness ($R_{ND} = R_a/D$) equal to 4.5 10^{-3}. On the other hand, crown splash occurs with ethanol at K = 2085 and $R_{ND} = 9.28\ 10^{-3}$. This implies that the Weber and Ohnesorge numbers do not define completely the physical problem through the Buckingham Theorem, other physical parameters should also be considered.

In the final phase of the spreading, the drop may begin to recede. A parameter, particularly significant for this process, is the receding contact angle. After reaching the maximum radius, if no crown has appeared, the lamella oscillates and moves towards the equilibrium position. The equilibrium shape is determined by the static contact angle and secondarily by the drop volume. On completely wettable surfaces ($\theta = 0^0$) the drop will continue to spread, and by definition a maximum spreading radius has clearly no meaning. In reality, drop impact on very wettable surfaces still produces an asymptotic value for the lamella radius because the shrinking due to liquid evaporation compensates the slow wetting. If the surface is not wettable, the lamella will recoil. A new phenomenon, observed by Rioboo et al. (2000a,2000b), has been defined as the "plateau" phase. The spreading diameter stalls for some time at an almost constant value before continuing to spread. The "plateau" phase is followed by the *pure wetting phase* for completely wettable systems.

During the recoil the lamella can break. As the liquid recedes, its dynamical contact angle becomes smaller. As the limiting value of zero is reached, some drop is left behind by the receding lamella. The lamella can even form a central jet or rebound from the surface, depending on the available excess of energy [Mao et al. (1997)]. High impact velocities lead to a larger diameter of the spreading film, which, in the receding phase, is more likely to break-up.

Zhang and Basaran (1997) studied carefully the effect of a surfactant during the spreading and recoiling phases of an impact. The dynamics is influenced by the presence of an extra surface tension term in the force balance. The maximum

spreading radius of surfactant-laden droplets is found to be smaller than with pure liquid droplets in the case of water.

3 Numerical Studies

Droplet-wall interactions offer a striking challenge to numerical and theoretical methods The simplest models involve inviscid fluid mechanics alone [Weiss and Yarin (1999)]. These models predict correctly several features of the splashing phenomenon. Inviscid fluid dynamics may be computed using the boundary integral method (BIM), in which the interface is tracked by a series of N markers and an integral equation is solved for the strength of sources along the interface. Axial symmetry is most often used and the fluid equations are then two dimensional, with a one-dimensional interface. The integral equation to be solved is then one-dimensional and the amount of computation required is of the order of N^2.

It is often desirable to take fluid viscosity into account. Then the Navier-Stokes equations must then be solved, with either a single fluid or two fluids, when the flow in the outside gas is important and thus it must be computed. These equations are supplemented with conditions for the continuity of velocity and stresses along the interface. As in the other formulation, a description of contact line motion is also needed when the impact is on a solid surface. Rather few studies tackle contact line motion, with the recent exception of Renardy (2001), and we shall thus leave this aspect of the problem somewhat aside. Many methods exist for the solution of the Navier-Stokes equations with interfaces. One of the aspects in which the methods differ widely is the treatment of interface reconnections. Such a reconnection may occur in various ways in the droplet impact problem: the most important one is the instant of impact of the droplet on the fluid layer, but later on during the development of the phenomenon, reconnections may also occur, for instance when secondary droplets are formed. It is thus important to reflect on the physical origins of reconnection. In some cases, reconnection is related to a singularity in finite time of the Navier-Stokes equations [Eggers (1997)], related to the Savart-Plateau-Rayleigh instability of jets. In other cases, van der Waals forces between the two interfaces attract them causing the reconnection to occur. Thus a realistic model of the reconnection should also take into account microscopic forces between interfaces. Molecular forces between the interface and the solid surface may also play a role.

Beyond reconnection other physical effects may be of importance: surface tension varying in space and time, surfactant molecules transport, visco-elastic effects and the finite thickness of the interface. However, it is important, when considering the challenge posed by droplet impact and various increases in realism and complexity, to know where to stop. Thus we shall mostly restrict this short review to inviscid and Navier-Stokes approaches with constant surface tension.

Among numerical methods, most of the methods applicable to interfaces in general, reviewed by Scardovelli and Zaleski (1999), are also applicable to splashing. Fukai et al. (1993,1995) used finite element methods with moving grids. However a larger number of authors used methods in which the Navier-Stokes equations are solved on a fixed, regular grid of size N There are three main fixed-grid methods: marker particle chains, volume of fluid (VOF) and level sets. All three methods in a two-dimensional space require of the order of N x N operations, but so does the boundary integral method (BIM), thus there is no obvious computational advantage in using an inviscid, BIM formulation. Moreover our own comparisons of both methods show that the CPU time required by the methods is also comparable in practice.

However, the Navier-Stokes formulation is limited in Reynolds number by the

grid resolution. Thus, using a BIM method could be a way to attempt to reach higher Reynolds numbers. However, it is well known that the inviscid formulation of fluid mechanics is not necessarily the limit of the solutions of the Navier Stokes equations at infinite Reynolds number.

The numerical modeling of interfaces may be roughly divided in three parts: the methods used to solve the Navier-Stokes equation in the absence of interface, the "geometric" or "kinematic" tracking of the interface and all the other aspects of the dynamics of the interface, such as surface tension, stress conditions or effects of density jumps. Among fixed grid methods, VOF methods are particularly attractive. The topology of the interface is not fixed by the representation of the interface data. Instead, the data (the information the algorithm has at the beginning of a time step) is the volume fraction occupied by the reference fluid in a given cell. The VOF algorithm then tries to guess what the shape of the interface is and then propagates it.

The interface may be reconstructed by straight lines parallel to the grid axis as in first order methods. However, it is found that to obtain sufficiently accurate and stable codes, it is necessary to use at least second order methods, where the interfaces are represented by straight lines of arbitrary orientation (or portion of planes in the three-dimensional space). The third dynamical aspect (the correct representation of surface tension and jump or free surface conditions) is another important issue in fixed grid methods. In the current state of the art, the dynamical issue is at least as important, or even more important, than the issue of correct tracking of the interface.

Results obtained by numerical calculations are mostly qualitative. Figure 8 shows an axis-symmetric VOF calculation performed using the VOF code of LMM [Gueyffier et al. (1999)]. A droplet, with a diameter of 4 mm, impacts on liquid film 0.9 mm thick, with a velocity of 20 cm/s. The density ratio is 500, and the two fluids are 10 times more viscous than air and water. As a result the Weber number is 4. The evolution of the interface showing the formation of the crown is depicted in the figure.

One of the striking features of these simulations [Gueyffier (2000)] is the fact that the jet starts very early. Figure 9 shows the early development of the jet. The more refined the simulation is, the earlier we observe the jet.

Another interesting point is the development of the corolla. At intermediate times it bends slightly backwards, but then it tips forward until it is almost vertical. At late times, the base of the corolla is vertical, while the top bends forwards. The fact that the top bends forward may probably be explained by the inertia of the top part of the corolla and the end rim.

This whole picture differs from the potential jet-impact theory, which would predict a corolla bending strongly backwards at late times. However it is possible than in these simulations a true late time has not yet been attained. The spreading of the corolla follows approximately the R $t^{1/2}$ law, but with a slowing down near the end of the simulation. This slowing down disagrees with the standard theory but agrees with the experiments.

Three-dimensional simulations have also been performed by several authors [Gueyffier and Zaleski (1998), Rieber and Frohn (1998,1999), Bussman et al. (2000)]. To obtain splashing impacts, the authors need to add "by hand" small perturbations to the droplet surface or to the liquid layer surface. The corrugations and instabilities that develop may be due either to the Richtmyer-Meshkov instability of the accelerated jet at early times [Gueyffier and Zaleski (1998)], or the Savart-Plateau-Rayleigh instability of the end rim [Rieber anf Frohn (1998)]. Additional information may be found on the LMM web site at http://www.lmm.jussieu.fr/ zaleski/gouttes.html

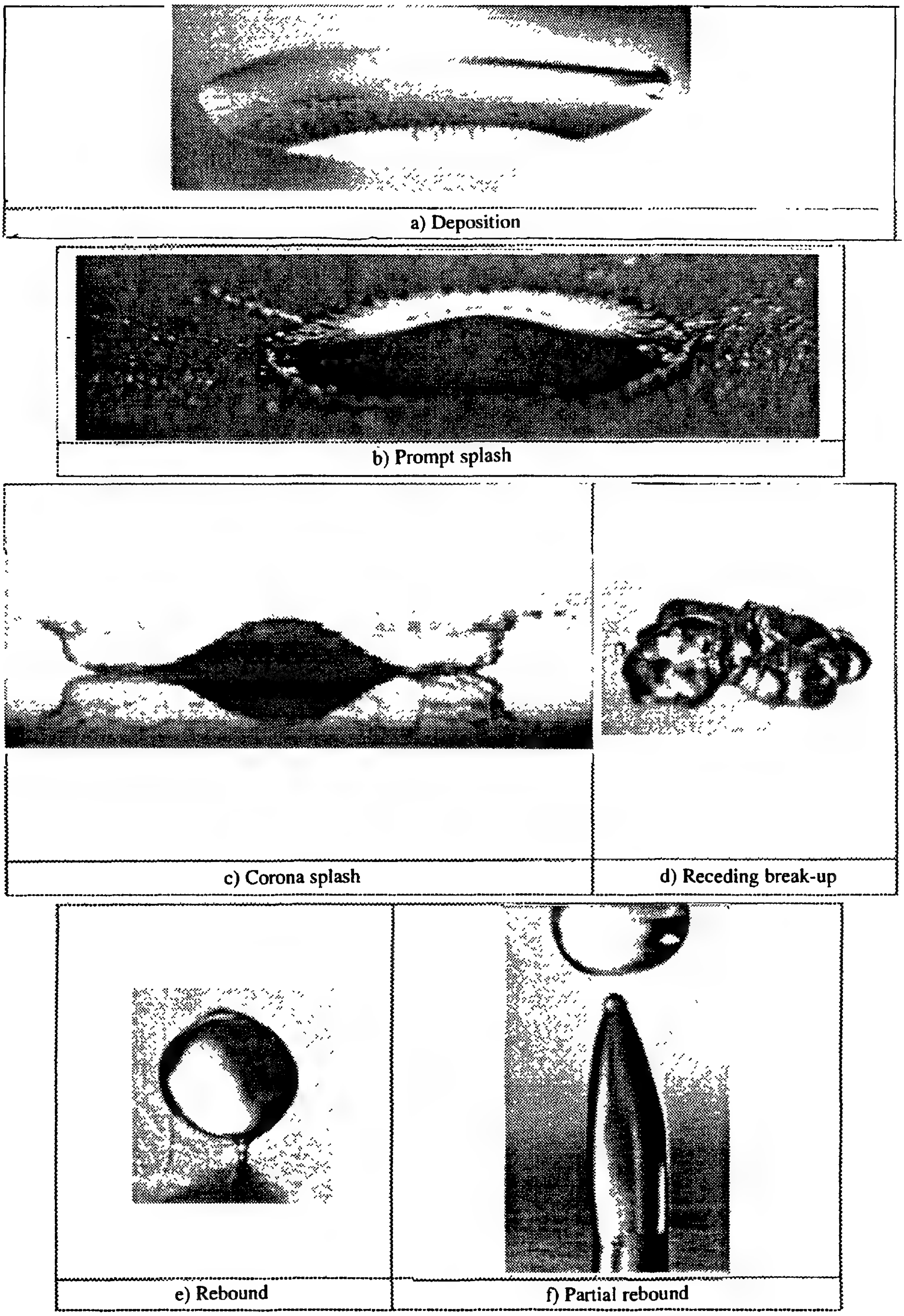


Figure 1: Examples for the 6 kinds of outcomes (liquid, surface, velocity, advancing/receding contact angle): a) water, smooth glass, 1.16 m/s; 10°/6°; b) water, rough glass, 3.6 m/s, 10°/6°; c) Isopropanol, smooth glass, 2.89 m/s, 0°/0°; d) water, smooth wax, 3.6 m/s, 105°/95°; e) water, rough wax, 1.18 m/s, 105°/85°; f) water, smooth wax, 1.18 m/s, 105°/95°.

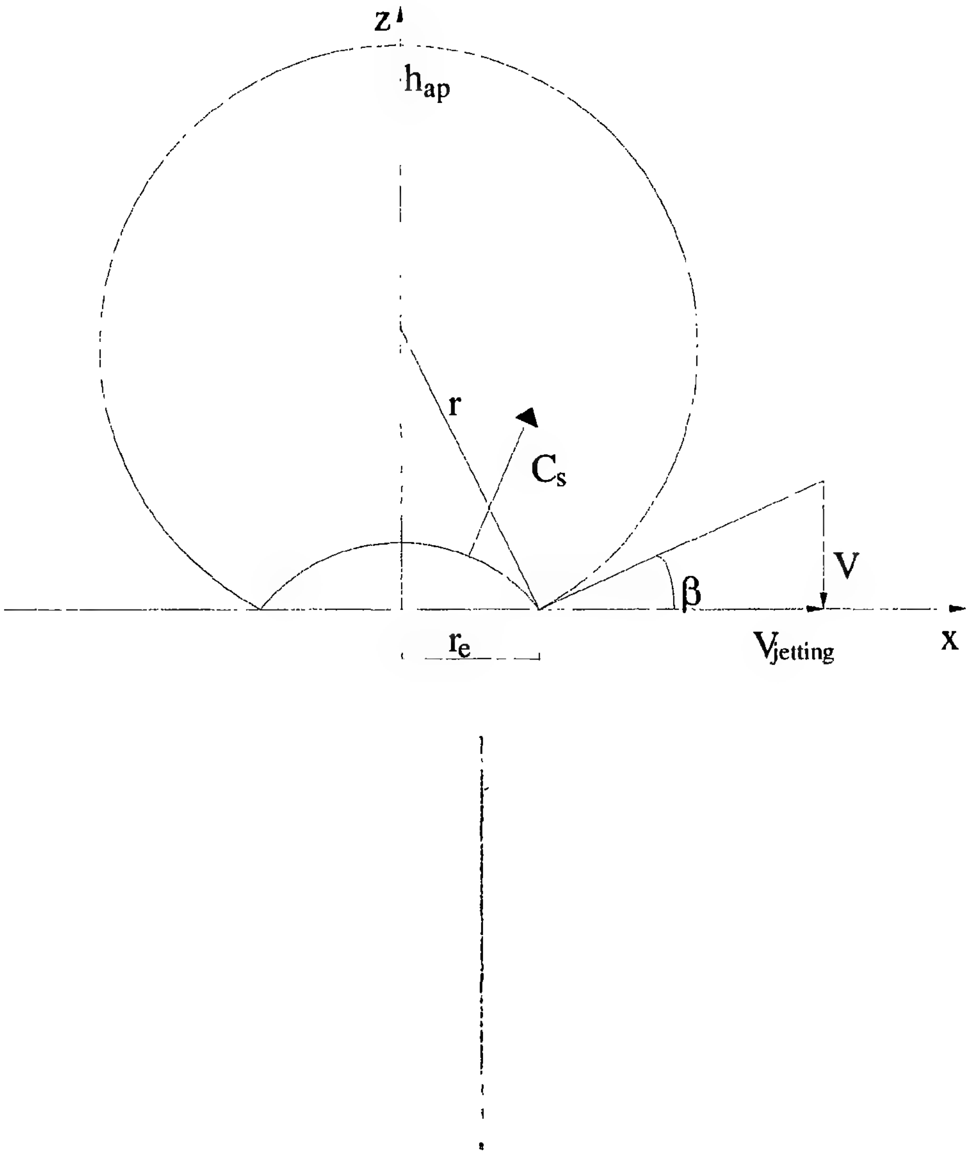


Figure 2: Shock wave formation and lamella ejection in the first impact instants. V: impact velocity, C_s: sound velocity, r: drop radius, r_e: contact radius, β: angle between the solid surface and the drop interface, $V_{jetting}$: velocity of lamella ejection.

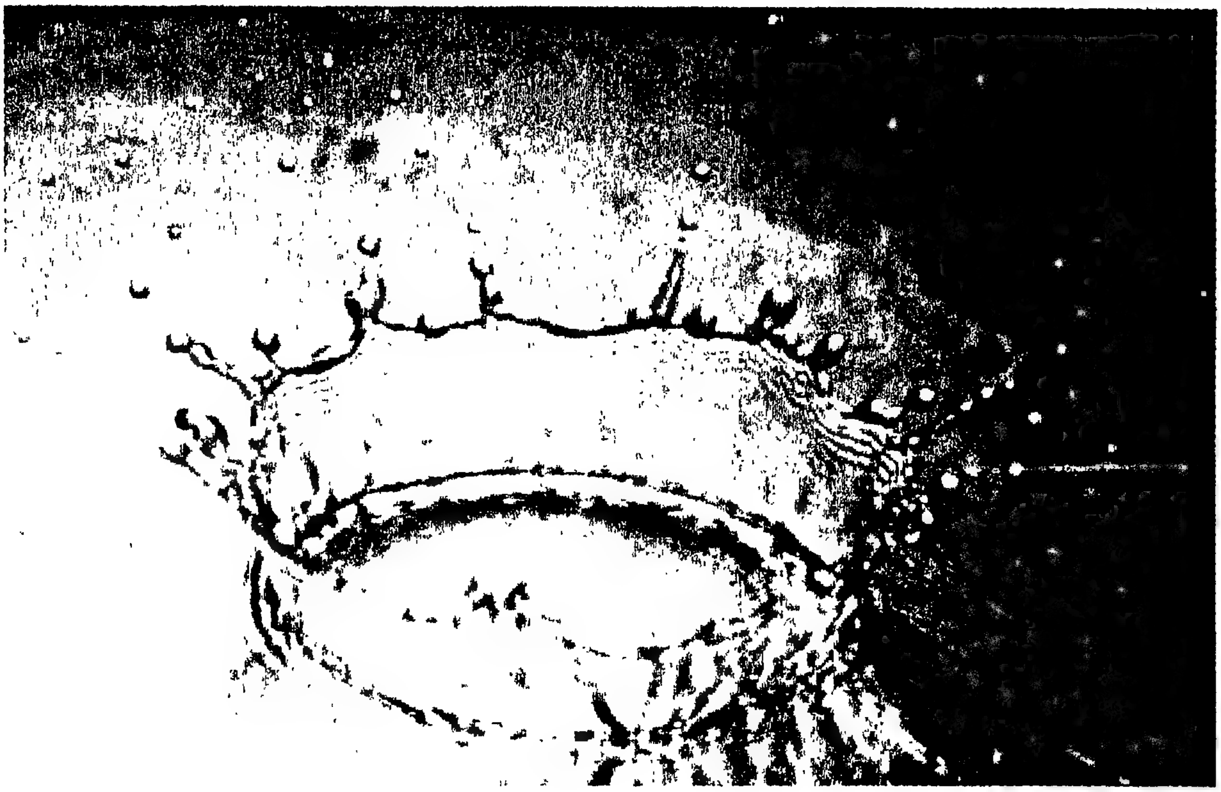


Figure 3: Drop impact on a liquid film. We = 560; Oh = $2 \cdot 10^{-3}$, $\delta = 0.1$, t = 8.3 ms.

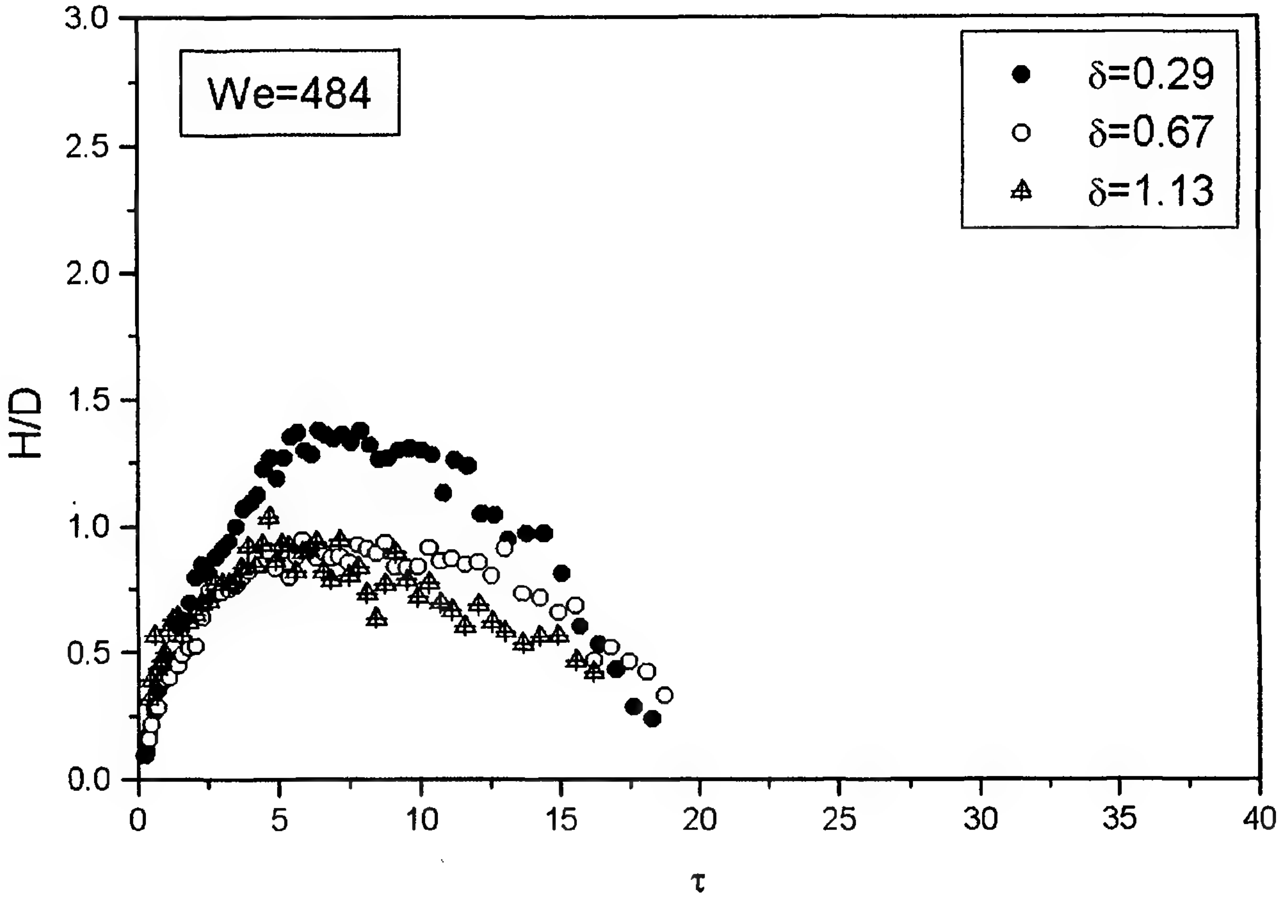
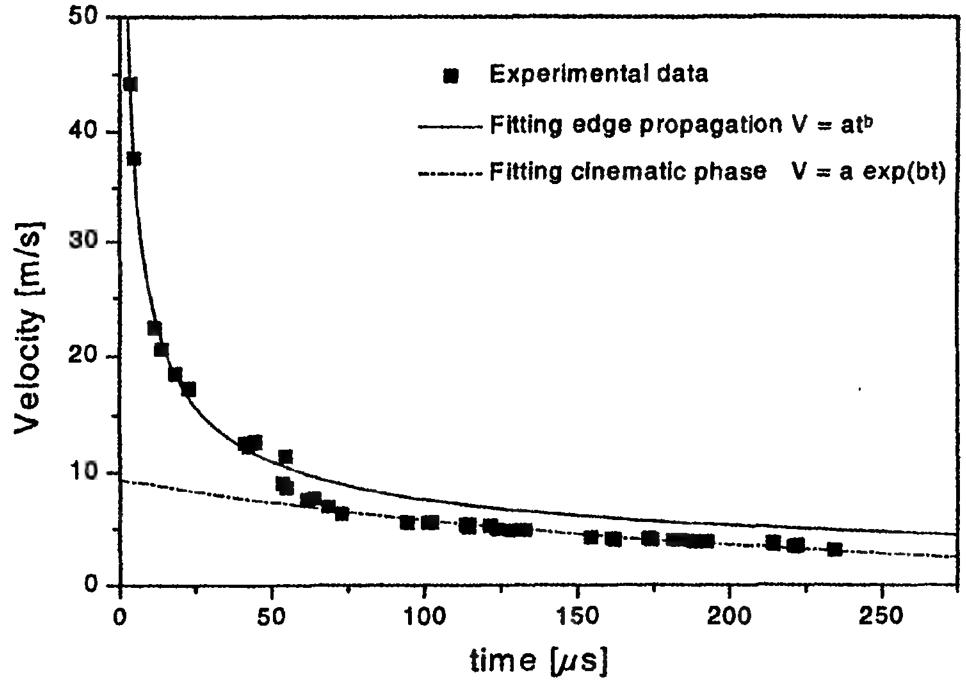
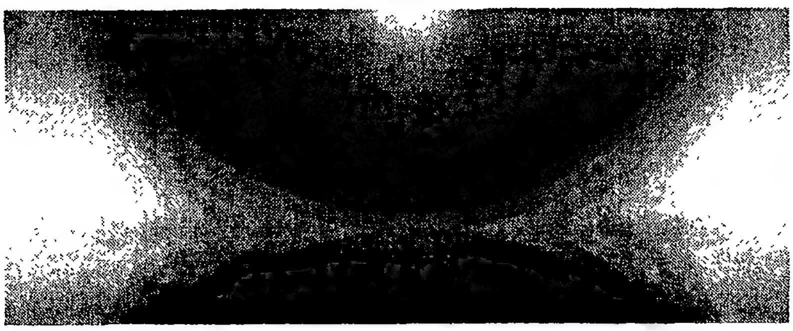


Figure 4: Crown height evolution for waterdrop impact on wetted surface.



(a)



(b)

Figure 5: (a) Evolution of the contact area diameter after a waterdrop impact onto a smooth PVC surface (We=88). (b) Multiframe image of a drop impact. The lamella extension can be directly evaluated together with its velocity. Δt=60 μs;t<250 μs

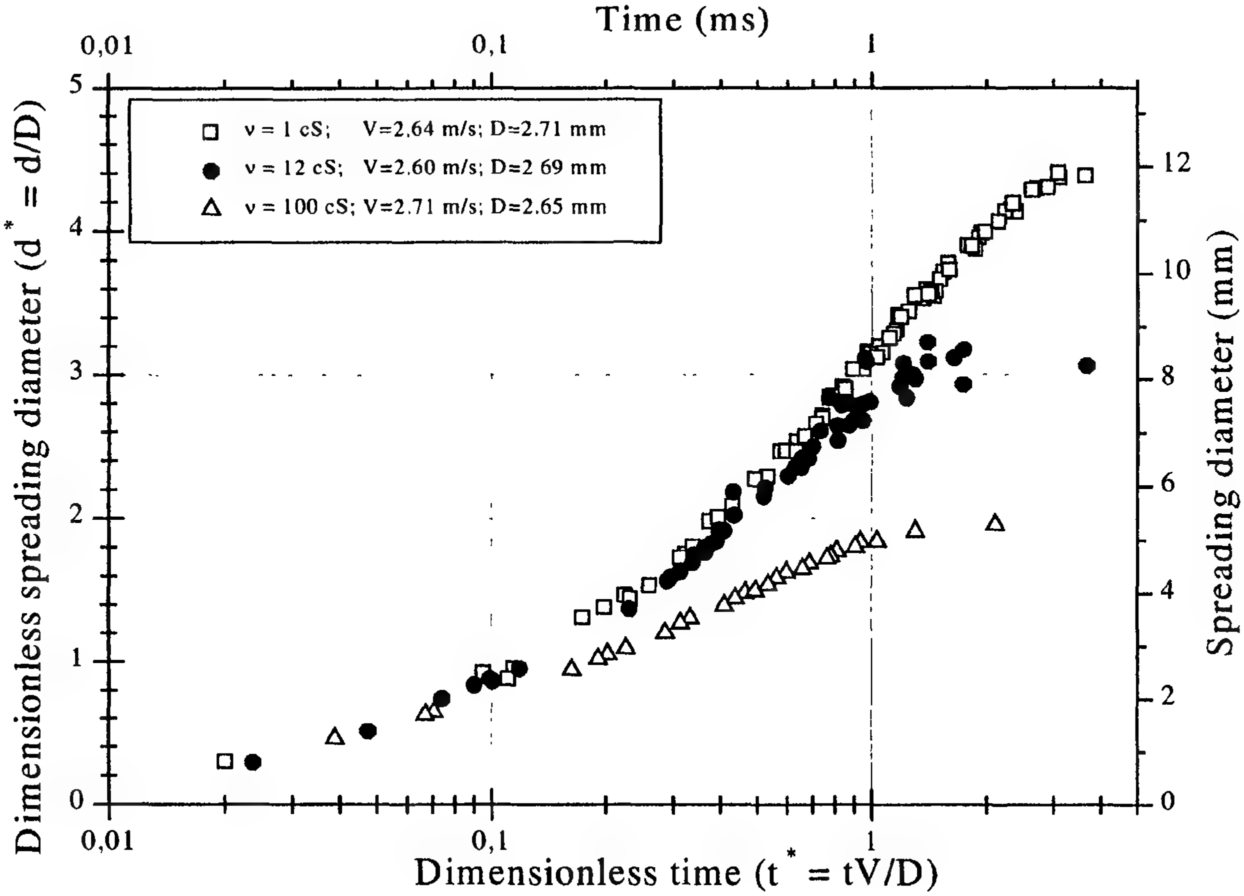


Figure 6: Influence of the viscosity on the spreading diameter in time. Dimensional and dimensionless forms can be seen on the same graph since V/D is held constant.

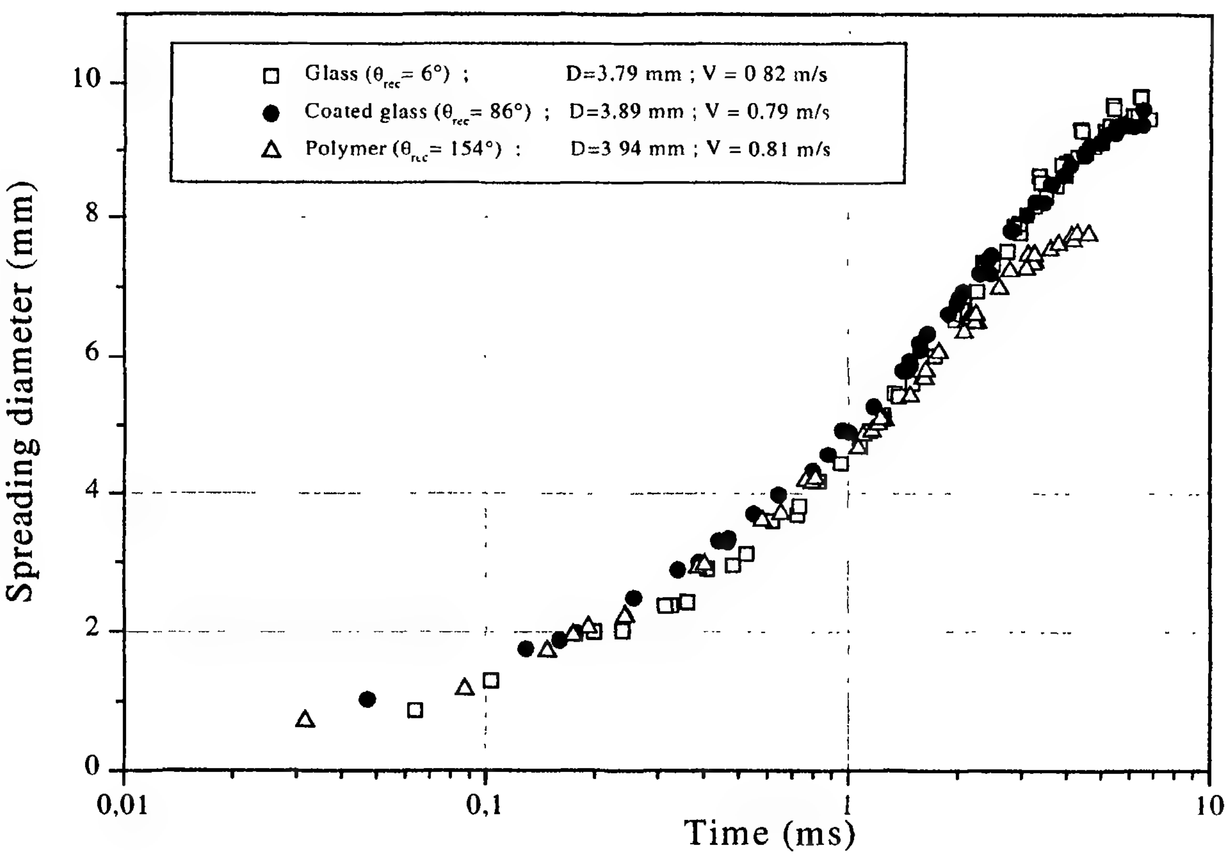


Figure 7: Wettability influence on the spreading behavior. All experiments are made with water. The drops sizes and velocities are comparable.

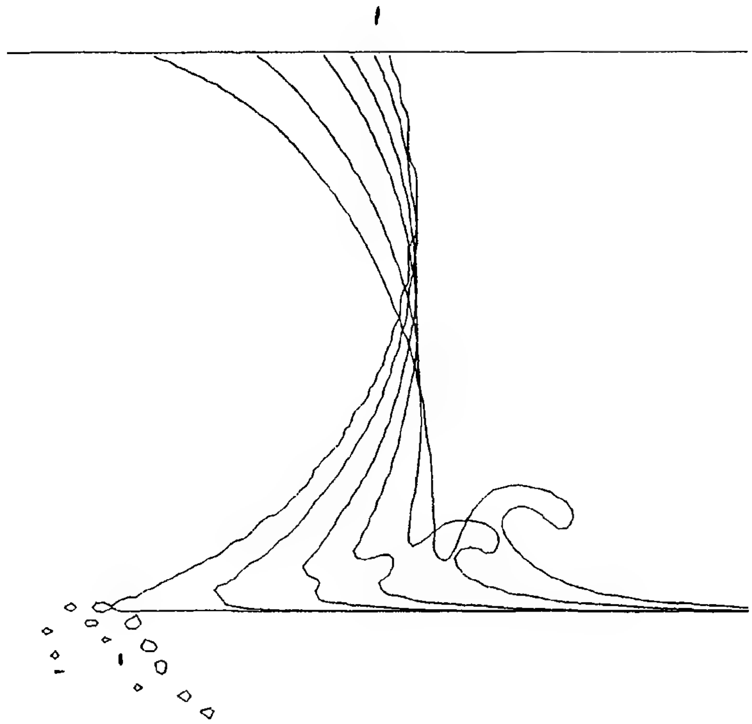


Figure 8: Early stages of the droplet impact on a liquid film.

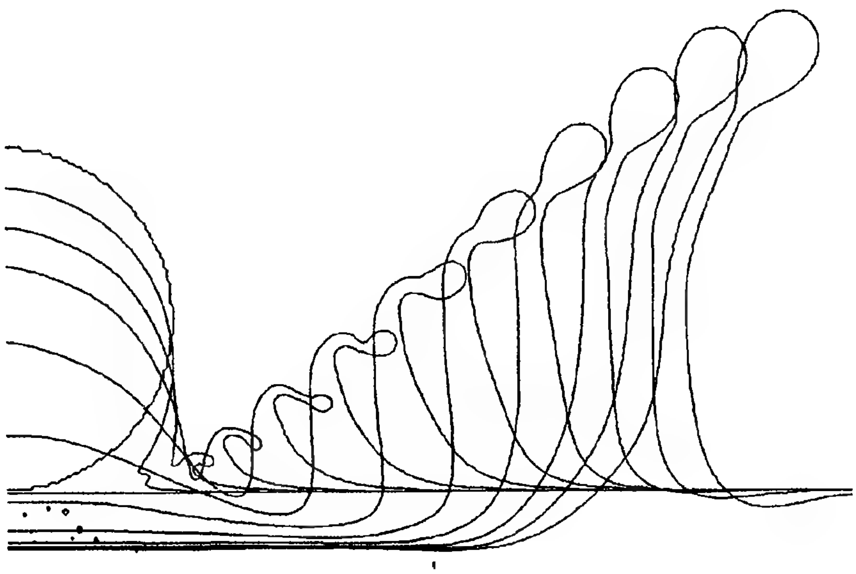


Figure 9: A series of interface profiles for a droplet impact. The Weber number is 4, other parameters are in the text.

Regime			Parameter range	Limit	Corrections			
Normal impacts	Dry	Rough	$R_{nd} > 0.01$	$K_{cr} = 660$	*Low wettable*	If the surface has a low wettability, the limit is greater	*Low surface tension ($\sigma < 0.03$ N/m)*	The limit can be lower. The splash can have different features
		Smooth	$R_{nd} < 0.01$	$K_{cr} = 471 + 73R_{nd}^{-0.3}$				
	Wetted	Thin film	$L_{nd} < \delta < 3R_{nd}^{0.16}$	$K_{cr} = 579 + \dfrac{33.0 + 527\delta^{2.44}}{R_{nd}^{0.4} + 0.07\delta^{2.44}}$	*Wavy*	When the liquid surface is wavy, a reduction of K_{cr} up to 40% is advisable		
		Liquid film	$3R_{nd}^{0.16} < \delta < 1.5$	Water, Oh < 0.01 $K_{cr} = 2074 + 870\delta^{0.23}$ $\mu > 0.05$, Oh > 0.01 $K_{cr} = 2164 + 7560\delta^{1.78}$				
		Shallow pool	$1.5 < \delta < 4$	$K_{cr} > 3200$				
		Deep pool	$\delta >> 4$	Asymptotic $K_{cr} = 8000$				
Inclined surfaces	Dry	$\alpha > 70°$	Same behavior as a normal impact		**Notes** The surface is at ambient temperature ($T < T_{Nu}$). The surface curvature is negligible. Surfaces and liquids are not electrically charged. The liquid is Newtonian. The liquid on the surface is the same of the primary drop. The impacting drop is spherical.			
		$\alpha < 20°$	Rebound limit					
	Wetted	$\alpha > 60°$	Same behavior as a normal impact					
		$\alpha < 6°$	Rebound limit					

Table 1 Resume of experimental correlations for isothermal drop impact. When $K > K_{cr}$ splash occurs. T_{Nu} is the Nukiyama temperature

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Convergence of the interface in the finite element approximation for two-fluid flows

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1 Introduction

Many physically interesting phenomena in industrial processes and nature involve flow problems with interfaces or fronts. Examples of such phenomena include the casting or mold filling processes, the dendritic solidification and the crystal growth, etc. These phenomena consist of free surface flows and free interface flows. In the free surface flows which describe the interaction between a fluid and air, it is often that the flow equations are solved only for the fluid and zero traction is assumed on the interface. If we formulate them by *true* two-fluid flow model, we can also regard them as the free interface flow problems. Anyway, in these problems the surface tension effect also plays a fundamental role. It is presented at the fluid-fluid interface for immiscible media.

In this paper we propose the finite element method for two-fluid flows with free interface including surface tension effect. The numerical strategy to deal the interface is Eulerlian. The feature of this approach is very easy to construct the algorithm due to the availability of the fixed mesh, however, we can only capture the interface implicitly. In our approach the interface is considered as the 0 level set of the pseudo-density function introduced by Dervieux and Thomasset [3]. As for the numerical treatment of the surface tension effect, it is interpreted as a body force spread across the interfacial region with a finite thickness, which is slightly different from the continuum surface force model (CSF) developed by Brackbill et al [1].

On the other hand, the another important problem in the numerical simulation for two-fluid flows or two-phase flows is the numerical *reproduction* of the interface. Especially, in our approach it is difficult to discuss the convergence of the

approximate interface directly. In this paper we show the convergence of the approximate interface by using the Borel measure and the Heaviside operator under the assumption that the finite element scheme for the pure advection equation of the pseudo-density function has the reasonable accuracy.

An overview of this paper is as follows. In Section 2 we present our mathematical model. Section 3 is devoted to the modeling of the interface. In Section 4 we give the finite element scheme for two-fluid flow system including the surface tension effect. In Section 5 we define the finite element scheme for the pure advection equation and show the stability and the convergence of it. In Section 6, we discuss the convergence of the interface. Finally we present some numerical results in Section 7.

Let us now establish the notation used throughout this paper. Let Ω be a bounded domain in $\mathbf{R}^2$ with Lipschitz boundary Γ. For a nonnegative integer m, let $H^m(\Omega)$ be the standard Sobolev spaces and denote their norms and the semi-norms by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. We shall denote $H^0(\Omega)$ by $L^2(\Omega)$, and the norm and the inner product of $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. For functions defined on the cylinder $Q_T = \Omega \times (0,T)$, we shall also introduce some additional notations. Namely, for any Banach space X and $1 \le p < \infty$ let $L^p(0,T;X)$ be the space of all $X-$valued functions which are defined on $(0,T)$, measurable and

$$\|u\|_{L^p(0,T;X)} = (\int_0^T \|u(t)\|_X^p \, dt)^{1/p} < +\infty.$$

Simiarly we define $L^\infty(0,T;X)$. In particular, we denote $L^p(0,T;L^p(\Omega))$ by $L^p(Q_T)$, and $L^\infty(0,T;L^\infty(\Omega))$ by $L^\infty(Q_T)$, respectively.

2 Mathematical model

Let Ω be a bounded domain of $\mathbf{R}^2$ with the boudary Γ. The domain Ω consists of two time-dependent subdomain $\Omega_i(t), i = 1,2$ such that

$$\overline{\Omega} = \overline{\Omega_1(t)} \cup \overline{\Omega_2(t)} \quad \text{and} \quad \Omega_1(t) \cap \Omega_2(t) = \emptyset \ \text{ for } 0 \le t < \infty.$$

Each subdomains are filled by the *fluid#1* and *fluid#2*, respectively. We assume that two fluids are both viscous, incompressible and immiscible. The governing equations for unsteady, viscous, incompressible, immiscible two-fluid system and the incompressibility condition are given by the following :

$$\begin{cases} \rho_\alpha(\dfrac{\partial u_i^{(\alpha)}}{\partial t} + u_j^{(\alpha)}\dfrac{\partial u_i^{(\alpha)}}{\partial x_j}) = \dfrac{\partial}{\partial x_j}\sigma_{ij}^{(\alpha)} - \rho_\alpha g \delta_{i2} & \text{in } \Omega_\alpha(t) \times (0,T), \\ & \text{for } i = 1,2 \text{ and } \alpha = 1,2, \\ div\ u^{(\alpha)} = 0 \quad \text{in } \Omega_\alpha(t) \times (0,T), \end{cases} \tag{2.1}$$

where $u^{(\alpha)} = (u_1^{(\alpha)}, u_2^{(\alpha)})$be the velocity, $p^{(\alpha)}$ the the pressure, μ_α the viscosity, ρ_α the density, g the gravitational acceleration. Here the stress tensor is defined by

$$\sigma_{ij}^{(\alpha)} = -p^{(\alpha)}\delta_{ij} + 2\mu_\alpha D_{ij}(u^{(\alpha)}), \tag{2.2}$$

where

$$D_{ij}(u^{(\alpha)}) = \frac{1}{2}(\frac{\partial u_i^{(\alpha)}}{\partial x_j} + \frac{\partial u_j^{(\alpha)}}{\partial x_i}). \tag{2.3}$$

Here we have used the summation convention.

As for the boundary condition and the initial condition, we assume the no-penetration condition on solid walls :

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.4}$$

and

$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{in } \Omega, \tag{2.5}$$

where $\boldsymbol{u}_0$ is a prescribed divergence-free velocity, respectively.

Furthermore we assume the transmission conditions at the interface between two fluids. One is the the continuity of the velocity :

$$\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} \quad \text{on } I(t), \tag{2.6}$$

and other is the mechanical condition :

$$\sigma_{ij}^{(1)} n_j - \sigma_{ij}^{(2)} n_j = \sigma \kappa n_i \quad \text{on} \quad I(t), i = 1, 2, \tag{2.7}$$

where σ is the surface tension coefficient, κ the curvature of the interface $I(t)$ and $\boldsymbol{n}$ the unit outward normal vector along the interface $I(t)$. In what follows we assume that the surface tension coefficient σ is a constant.

3 Advection of the interface

By the assumption for the immiscibility of two fluids, the motion of the interface is governed by

$$\boldsymbol{u}^{(\alpha)} \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n}, \tag{3.1}$$

where $\boldsymbol{u}^{(\alpha)}$ is the velocity of *fluid#α* on the interface for $\alpha = 1, 2$, $\boldsymbol{v}$ the speed of the interface displacement and $\boldsymbol{n}$ the unit normal vector to the interface.

If we consider that the interface $I(t)$ is defined as follows :

$$I(t) = \{\boldsymbol{x} = (x_1, x_2) | \ \varphi(\boldsymbol{x}, t) = 0 \text{ for } 0 \leq t < T\}, \tag{3.2}$$

it is easiliy seen that $\varphi = \varphi(\boldsymbol{x}, t)$ satisfy the following equation :

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi = 0 \qquad \text{in } \Omega \times (0, T). \tag{3.3}$$

In practical computation, we choose an sufficiently smooth initial function for the transport equation so that

$$\varphi_0 = \begin{cases} < 0 & \text{in } \Omega_1(0), \\ \geq 0 & \text{in } \Omega_2(0). \end{cases} \tag{3.4}$$

Due to [3] the solution of (3.3)-(3.4) is called the *pseudo-density* function.

Remark 1. Recently, for the numerical analysis for two-phase problems it is well known that the *level set function* [2] is usefull, however, there is a slight difference between the pseudo-density function and the level set function. Initially, the level set fuction ϕ is set equal to the signed distance function from the interface. That is

$$\phi_0 = \begin{cases} +d & \text{in } \Omega_1(0), \\ 0 & \text{on } I(t), \\ -d & \text{in } \Omega_2(0), \end{cases} \tag{3.5}$$

where d is the distance from the interface $I(t)$. But the pseudo-density function does not have this property.

Using the pseudo-density function, we can obtain the outer normal unit vector and the curvature, respectively :

$$\boldsymbol{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \Big(\frac{\varphi_x}{\sqrt{\varphi_x^2 + \varphi_y^2}}, \frac{\varphi_y}{\sqrt{\varphi_x^2 + \varphi_y^2}}\Big), \tag{3.6}$$

$$\kappa = -\nabla \cdot \boldsymbol{n} = -\frac{\varphi_y^2\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + \varphi_x^2\varphi_{yy}}{(\varphi_x^2 + \varphi_y^2)^{3/2}}. \tag{3.7}$$

In order to get these quantities, we need to find accurate values of φ, φ_x, φ_y, φ_{xx}, φ_{xy}, φ_{yy}. This suggests that we have to use at least the finite element of 6 degrees of freedom to approximate the pseudo-density function by the finite element method.

4 Finite element scheme for two-fluid flows

We introduce the following space :

$$\begin{cases} \boldsymbol{V} = \{\boldsymbol{v} \in H^1(\Omega)^2 \,|\, \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}, \\ Q = L_0^2(\Omega) = \{q \in L^2(\Omega) \,|\, \int_\Omega q\,dx = 0\}. \end{cases} \tag{4.1}$$

The variational formulation for the gorverning equations (2.1) is obtained by multiplying (2.1) by an arbitrary function $v \in H^1(\Omega)$, integrating over the domain Ω_α and adding two equations.

$$\text{Find } \ (\boldsymbol{u}(t), p(t)) \in \boldsymbol{V} \times Q \ \text{ for } 0 < t < T$$

such that

$$\begin{cases} \int_\Omega \{\rho(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j})\} v dx - \int_\Omega p \frac{\partial v}{\partial x_i} dx + 2\mu \int_\Omega D_{ij}(u) \frac{\partial v_i}{\partial x_j} dx \\ = - \int_\Omega \rho g \delta_{i2} v_i dx + \int_{I(t)} \sigma \kappa n_i v_i d\gamma, \text{ for } \forall v = (v_1, v_2) \in V, i = 1, 2. \\ \int_\Omega q \ div \ u \ dx = 0 \quad \text{for } \forall \ q \in Q, 0 < t < T, \end{cases} \tag{4.2}$$

where for $\alpha = 1, 2$

$$\begin{cases} u = u^{(\alpha)} & \text{in } \Omega_\alpha(t), \\ p = p^{(\alpha)} & \text{in } \Omega_\alpha(t), \\ \rho = \rho_\alpha & \text{in } \Omega_\alpha(t), \\ \mu = \mu_\alpha & \text{in } \Omega_\alpha(t). \end{cases} \tag{4.3}$$

In order to approximate the variational formulation (4.2) by the finite element method, we construct the finite element space using $P1\, iso\, P2/P1$ element :

$$\begin{cases} V_h = \{v_h \in C^0(\overline{\Omega})^2 \mid v_h|_{\tilde{K}} \in P_1(\tilde{K})^2 \ \ \forall \tilde{K} \in \mathcal{T}_{h/2}, \ v_h \cdot n|_\Gamma = 0\}, \\ Q_h = \{q_h \in C^0(\overline{\Omega}) \mid q_h|_K \in P_1(K) \ \ \forall K \in \mathcal{T}_h\} \cap \ L_0^2(\Omega), \end{cases} \tag{4.4}$$

where $\{\mathcal{T}_h\}_{h>0}$ be a *regular* family of triangulations of $\overline{\Omega}$ and $\mathcal{T}_{h/2}$ be a new triangulation obtained by deviding each element $K \in \mathcal{T}_h$ into four equal subtriangles. As shown in (4.2), the surface tension effect is represented by the line integral on the interface $I(t)$. In this paper, the surface tension effect is interpreted as a body force spread across the interfacial region $\mathcal{R}_\mathcal{I}(t)$ with a finite thickness, which is defined by

$$\mathcal{R}_\mathcal{I}(t) = \bigcup_{\substack{K \cap I(t) \neq \emptyset \\ K \in \mathcal{T}_h}} K. \tag{4.5}$$

Therefore in the variational formulation (4.2) we replace

$$\int_{I(t)} \sigma \kappa n_i v d\gamma$$

by the volume integral on the interfacial region $\mathcal{R}_\mathcal{I}(t)$.

In order to define a full discretization of (4.2) we consider the uniform mesh for the time variable t and define

$$t_n = n\tau \ \text{ for } n = 0, 1, 2, \cdots, [T/\tau],$$

where $\tau > 0$ is a time-step. Then our approximate problem is defined as follows, where u_h^n, and p_h^n, denotes the approximation of the exact solution $u(x, t_n)$, and $p(x, t^n)$, respectively.

Find $(u_h^{n+1}, p_h^n) \in V_h \times Q_h$ such that

$$\begin{cases} \rho\{(\frac{u_h^{n+1} - u_h^n}{\tau}, v_h) + (u_h^n \cdot \nabla u_h^n, v_h)\} - (p_h^n, div\ v_h) + 2\mu(D_{ij}(u_h^n), D_{ij}(v_h)) \\ \qquad\qquad = (f, v_h) + (f_{ST}^n, v_h) \quad \text{for } \forall v_h \in V_h, \\ (div\ u_h^{n+1}, q_h) = 0 \quad \text{for } \forall q_h \in Q_h, \end{cases} \tag{4.6}$$

where $f = {}^t[0 \quad -\rho g]$ and

$$(f_{ST}^n, v_h) = \sum_{K \in R_I(t_n)} \int_K C\sigma\kappa n \cdot v_h \, dx. \tag{4.7}$$

In practice we implement the approximate problem (4.6) by the velocity explicit and pressure implicit scheme [4] using the mass *lumping* matrix, however, we omit it in this paper.

Remark 2. In the finite element scheme (4.6) with $P1\,iso\,P2$ element, we can take $C = 3\eta/h_K$ in (4.7), where h_K is the diameter of K and $\eta > 0$ is the constant derived from the regularity of the family of triangulations.

5 Finite element scheme for the transport equation

In order to decouple the nonlineality of the Navier-Stokes equations (2.1) and the transport equation (3.3), in practice we may consider the following *pure* advection problem instead of (3.3).

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + u^{n+1} \cdot \nabla \varphi = 0 & \text{in } \Omega \times [t_n, t_{n+1}), \\ & \text{for } n = 0, 1, \cdots, [T/\tau] - 1, \\ \varphi(x, 0) = \varphi_0 & \text{in } \Omega. \end{cases} \tag{5.1}$$

where $u^{n+1} = u(x, t_{n+1})$ and φ_0 is the sufficiently smooth function defined by (3.5). In the sequel, we assume that for $n = 0, 1, \cdots, [T/\tau] - 1$

$$\begin{cases} u^{n+1} \in L^\infty(\Omega), \\ div\ u^{n+1} = 0 & \text{in } \Omega, \\ u^{n+1} \cdot n = 0 & \text{on } \Gamma. \end{cases} \tag{5.2}$$

Then we consider an implicit Euler scheme for the pure advection equation (5.1), which is the discretization in time and the finite element approximation is adopted in space variable using the finite element of 6 degrees of freedom at least. Let V be a Sobolev space $H^1(\Omega)$ and V_h be the finite dimensional space of V such that

$$V_h = \{v_h \in C^0(\overline{\Omega}) \mid v_h|_K \in P_k(K) \text{ for } \forall K \in \mathcal{T}_h\}, \tag{5.3}$$

where $k \geq 1$. For $1 \leq l \leq k$ let Π_h be the interpolation operator from $H^{l+1}(\Omega)$ to V_h. It is well-known that

$$\|v - \Pi_h v\|_{0,\Omega} + h|v - \Pi_h v|_{1,\Omega} \leq Ch^{l+1}|v|_{l+1,\Omega} \quad \text{for } \forall v \in H^{l+1}(\Omega). \tag{5.4}$$

Then our finite element scheme for (5.1) is as follows :

$$\text{Find } \varphi_h^{n+1} \in V_h \text{ such that}$$

$$\begin{cases} (\dfrac{\varphi_h^{n+1} - \varphi_h^n}{\tau}, v_h) + (u^{n+1} \cdot \nabla \varphi_h^{n+1}, v_h) = 0 \quad \text{for } \forall v_h \in V_h, \\ \varphi_h^0 = \varphi_{0h}, \end{cases} \tag{5.5}$$

where $\varphi_{0h} = \Pi_h \varphi_0$.

Remark 3. In practice we use the P_1 *iso* $P2$ element for the scheme (5.1). For this element we have the following estimate :

$$\|v - \Pi_h v\|_{0,\Omega} + h|v - \Pi_h v|_{1,\Omega} \leq Ch^2|v|_{2,\Omega} \quad \text{for } \forall v \in H^2(\Omega). \tag{5.6}$$

As for the stability and the convergence of the finite element scheme (5.5), we have the following results.

Proposition 1. For each $n = 0, 1, 2, \cdots, [T/\tau]$ it holds

$$\|\varphi_h^n\|_{0,\Omega} \leq \|\varphi_{0h}\|_{0,\Omega}. \tag{5.7}$$

Theorem 1. Assume that $\varphi_0 \in H^k(\Omega)$ and the solution φ to (3.3) satisfies

$$\varphi \in L^2(0,T; H^{k+1}(\Omega)), \ \frac{\partial \varphi}{\partial t} \in L^2(0,T; H^k(\Omega)) \text{ and } \frac{\partial^2 \varphi}{\partial t^2} \in L^\infty(Q_T). \tag{5.8}$$

Then φ_h^n defined by (5.5) satisfies for each $n = 0, 1, 2, \cdots, [T/\tau]$

$$\|\varphi_h^n - \varphi(\cdot, t_n)\|_{0,\Omega} \leq C(h^k + \tau), \tag{5.9}$$

where $C > 0$ is a constant.

Proof. We consider the semi-discretization in time for the equation (5.1).

$$\begin{cases} \dfrac{\varphi^{n+1} - \varphi^n}{\tau} + u^{n+1} \cdot \nabla \varphi^{n+1} = 0 \ \text{ in } \Omega, \\ \varphi^0 = \varphi_0. \end{cases} \tag{5.10}$$

Noticing that it holds from (5.1)

$$\frac{\partial \varphi}{\partial t}(x, t_{n+1}) = -u^{n+1} \cdot \nabla \varphi(x, t_{n+1}),$$

we have the following by means of the Taylor's theorem.

$$\frac{\varphi(x,t_{n+1})-\varphi(x,t_n)}{\tau}+u^{n+1}\cdot\nabla\varphi(x,t_{n+1})=-\frac{\tau}{2}\frac{\partial^2\varphi}{\partial t^2}(x,t_{n+1}-\theta\tau), \tag{5.11}$$

where $0<\theta<1$. Subtracting (5.11) from (5.10), we have

$$\frac{\xi^{n+1}-\xi^n}{\tau}+u^{n+1}\cdot\nabla\xi^{n+1}=\frac{\tau}{2}\frac{\partial^2\varphi}{\partial t^2}(x,t_{n+1}-\theta\tau) \tag{5.12}$$

where $\xi^{n+1}=\varphi^{n+1}-\varphi(x,t_{n+1})$.

Multiplying (5.12) by $\xi^{n+1}\in V$ and integrating on Ω, we have for some $t^*=t_{n+1}-\theta\tau$

$$\|\xi^{n+1}\|_{0,\Omega}^2=(\xi^n,\xi^{n+1})+\frac{\tau^2}{2}(\frac{\partial^2\varphi}{\partial t^2}(x,t^*),\xi^{n+1}). \tag{5.13}$$

Since $\xi^0=\varphi^0-\varphi(x,0)=0$, it is easily shown that

$$\|\xi^n\|_{0,\Omega}\le C\tau\|\frac{\partial^2\varphi}{\partial t^2}\|_{L^\infty(Q_T)}. \tag{5.14}$$

On the other hand, we consider the variational formulation for the semi-discretization in time (5.10) :

Find $\varphi^{n+1}\in V$ such that

$$\begin{cases}(\frac{\varphi^{n+1}-\varphi^n}{\tau},v)+(u^{n+1}\cdot\nabla\varphi^{n+1},v)=0 \quad \text{for } \forall v\in V,\\ \varphi^0=\varphi_0.\end{cases} \tag{5.15}$$

Setting $v=v_h$ in (5.15) and subtracting the full scheme (5.5) from it, then we have

$$(\varphi_h^{n+1}-\varphi^{n+1},v_h)+\tau(u^{n+1}\cdot\nabla(\varphi_h^{n+1}-\varphi^{n+1}),v_h)=(\varphi_h^n-\varphi^n,v_h). \tag{5.16}$$

Here we set $v_h=\varphi_h^{n+1}-\Pi_h\varphi^{n+1}$. Noticing that the following equality holds

$$\begin{aligned}\|\varphi_h^{n+1}-\Pi_h\varphi^{n+1}\|_{0,\Omega}^2 \;=\;& -(\Pi_h(\varphi^{n+1}-\varphi^n)-(\varphi^{n+1}-\varphi^n),\varphi_h^{n+1}-\Pi_h\varphi^{n+1})\\ &-\tau(u^{n+1}\cdot\nabla(\Pi_h\varphi^{n+1}-\varphi^{n+1}),\varphi_h^{n+1}-\Pi_h\varphi^{n+1})\\ &+(\varphi_h^n-\Pi_h\varphi^n,\varphi_h^{n+1}-\Pi_h\varphi^{n+1}),\end{aligned}$$

we have

$$\begin{aligned}\|\varphi_h^{n+1}-\Pi_h\varphi^{n+1}\|_{0,\Omega}^2 \;\le\;& \underbrace{|(\Pi_h(\varphi^{n+1}-\varphi^n)-(\varphi^{n+1}-\varphi^n),\varphi_h^{n+1}-\Pi_h\varphi^{n+1})|}_{(1)}\\ &+\underbrace{\tau|(u^{n+1}\cdot\nabla(\Pi_h\varphi^{n+1}-\varphi^{n+1}),\varphi_h^{n+1}-\Pi_h\varphi^{n+1})|}_{(2)}\\ &+\|\varphi_h^n-\Pi_h\varphi^n\|_{0,\Omega}\,\|\varphi_h^{n+1}-\Pi_h\varphi^{n+1}\|_{0,\Omega}.\end{aligned}$$

From the assumption $\frac{\partial \varphi}{\partial t} \in L^2(0,T;H^k(\Omega))$ the first term is estimated as follows :

$$\begin{aligned}(1) \quad &\leq \|\Pi_h(\frac{\partial \varphi}{\partial t}(\cdot, t_n+\theta\tau)) - \frac{\partial \varphi}{\partial t}(\cdot, t_n+\theta\tau)\|_{0,\Omega}\, \|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega} \\ &\leq c_1\tau h^k\, \|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega}.\end{aligned}$$

As for the second term, we can estimate as follows :

$$\begin{aligned}(2) \quad &\leq \tau\|u^{n+1}\|_{L^\infty(\Omega)}\, \|\nabla(\Pi_h\varphi^{n+1} - \varphi^{n+1})\|_{0,\Omega}\, \|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega} \\ &\leq \tau\|u^{n+1}\|_{L^\infty(\Omega)}\; c_2 h^k\, \|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega}.\end{aligned}$$

Therefore we have

$$\|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega} \leq (c_1 + c_2\tau\|u^{n+1}\|_{L^\infty(\Omega)})h^k\tau + \|\varphi_h^n - \Pi_h\varphi^n\|_{0,\Omega}. \quad (5.17)$$

Since $\varphi_h^0 - \Pi_h\varphi_0 = 0$, it is easily shown that

$$\|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega} \leq Ch^k. \quad (5.18)$$

From (5.4), (5.14) and (5.18) we have

$$\begin{aligned}\|\varphi_h^{n+1} - \varphi(\cdot, t_{n+1})\|_{0,\Omega} &\leq \|\varphi_h^{n+1} - \Pi_h\varphi^{n+1}\|_{0,\Omega} + \|\Pi_h\varphi^{n+1} - \varphi^{n+1}\|_{0,\Omega} \\ &\quad + \|\varphi^{n+1} - \varphi(\cdot, t_{n+1})\|_{0,\Omega} \\ &\leq O(h^k) + O(h^{k+1}) + O(\tau) \\ &= O(h^k + \tau).\end{aligned}$$

6 Convergence of the interface

In this section we consider the convergence of the approximate interface by means of the measure theory. Let E be a 1-dimensional Borel set and $f = f(x)$ be a 1-dimensional Borel measurable function. We denote $meas\{x \in \Omega | f(x) \in E\}$ by $m[f \in E]$. Furthermore, we denote the Heaviside operator by $H(\cdot)$.

Lemma 1. Let $p \geq 1$ be a integer and $\delta > 0$. Then it holds that for any $\psi_1, \psi_2 \in L^2(\Omega)$

$$\|H(\psi_1) - H(\psi_2)\|_{L^p(\Omega)}^p \leq m[|\psi_1| < \delta] + m[|\psi_2| < \delta] + m[|\psi_1 - \psi_2| \geq 2\delta]. \quad (6.1)$$

Proof. Due to the definition of $H(\cdot)$, we have for some constant $\delta > 0$

$$\|H(\psi_1) - H(\psi_2)\|_{L^p(\Omega)}^p = m[\psi_1 \geq 0, \psi_2 < 0] + m[\psi_1 < 0, \psi_2 \geq 0]$$

$$\begin{aligned}
&= m[\psi_1 \geq \delta, \psi_2 \leq -\delta] + m[\psi_1 \geq \delta, -\delta < \psi < 0] \\
&\quad + m[\delta > \psi_1 \geq 0, \psi_2 \leq -\delta] + m[\delta > \psi_1 \geq 0, -\delta < \psi_2 < 0] \\
&\quad + m[\psi_1 \leq -\delta, \psi_2 \geq \delta] + m[\psi_1 \leq -\delta, \delta > \psi_2 \geq 0] \\
&\quad + m[-\delta < \psi_1 < 0, \psi_2 \geq \delta] + m[-\delta < \psi_1 < 0, \delta > \psi_2 \geq 0] \\
&= m[\psi_1 \geq \delta, \psi_2 \leq -\delta] + m[\psi_1 \leq -\delta, \psi_2 \geq \delta] \\
&\quad + m[\psi_1 \geq 0, -\delta < \psi_2 < 0] + m[\psi_1 < 0, 0 \leq \psi_2 < \delta] \\
&\quad + m[\delta > \psi_1 \geq 0, \psi_2 \leq -\delta] + m[-\delta < \psi_1 < 0, \psi_2 \geq \delta] \\
&\leq m[\psi_1 \geq \delta, \psi_2 \leq -\delta] + m[\psi_1 \leq -\delta, \psi_2 \geq \delta] + m[|\psi_1| < \delta] \\
&\quad + m[|\psi_2| < \delta] \\
&\leq m[|\psi_1| < \delta] + m[|\psi_2| < \delta] + m[|\psi_1 - \psi_2| \geq 2\delta].
\end{aligned}$$

The following lemma can be easily shown by the Schwartz inequality and the Chebyshev inequality.

Lemma 2. Let $\delta > 0$ be a constant. For any $\psi_1, \psi_2 \in L^2(\Omega)$ it holds that

$$m[|\psi_1 - \psi_2| \geq 2\delta] \leq \frac{1}{4\delta^2} \|\psi_1 - \psi_2\|_{0,\Omega}^2, \tag{6.2}$$

Theorem 2. Assume that $\varphi_0 \in H^k(\Omega)$ and the solution φ of (3.3) satisfies

$$\varphi \in L^2(0,T; H^{k+1}(\Omega)), \ \frac{\partial \varphi}{\partial t} \in L^2(0,T; H^k(\Omega)) \text{ and } \frac{\partial^2 \varphi}{\partial t^2} \in L^\infty(Q_T). \tag{6.3}$$

Furthermore, we assume that for sufficiently small $\delta > 0$ and $h > 0$ there exist constants $C_1 > 0$, $C_2 > 0$ independent of δ and h such that

$$\begin{cases} \sup_{h>0} m[|\varphi_h^n| < \delta] \leq C_1 \delta, \\ m[|\varphi(\cdot, t_n)| < \delta] \leq C_2 \delta. \end{cases} \tag{6.4}$$

If there exists a constant $C_3 > 0$ such that

$$\frac{\tau}{h^k} \leq C_3, \tag{6.5}$$

then we have for some constant $C = C(p) > 0$

$$\|H(\varphi_h^n) - H(\varphi(\cdot, t_n))\|_{L^p(\Omega)} \leq C h^{2k/3p}. \tag{6.6}$$

Proof. From Lemmas 1, 2 and Theorem 1 we have

$$\|H(\varphi_h^n) - H(\varphi(\cdot, t_n))\|_{L^p(\Omega)}^p \leq m[|\varphi_h^n| < \delta] + m[|\varphi(\cdot, t_n)| < \delta] + \frac{C^2}{4\delta^2}(h^k + \tau)^2.$$

Here we take $\delta = h^{2k/3}$. So that we have

$$\begin{aligned}\|H(\varphi_h^n) - H(\varphi(\cdot,t_n))\|_{L^p(\Omega)}^p &\leq (C_1 + C_2)h^{2k/3} + \frac{C^2}{2}h^{-4k/3}(h^{2k} + \tau^2) \\ &= (C_1 + C_2 + \frac{C^2}{2})h^{2k/3} + \frac{C^2}{2}h^{-4k/3}\tau^2.\end{aligned}$$

Since $\tau \leq C_3 h^k$, we have

$$\|H(\varphi_h^n) - H(\varphi(\cdot,t_n))\|_{L^p(\Omega)}^p \leq (C_1 + C_2 + \frac{C^2}{2} + \frac{C^2 C_3^2}{2})h^{2k/3}.$$

Remark 4. In the case of $P1$ *iso* $P2$ element, we have from Remark 3

$$\|H(\varphi_h^n) - H(\varphi(\cdot,t_n))\|_{L^p(\Omega)} \leq C\, h^{2/3p}. \tag{6.7}$$

7 Numerical results

In this section we present some numerical results to demonstarate the efficiency of our scheme. Rayleigh-Taylor instability is a fingering instability of the interface between two fluids of different densities, which occurs when a heavy fluid is superposed over a light fluid in a gravitational field acting downward. Any perturbation of this interface tends to grow with time, then the fluid interface is unstable. In this example we consider Rayleigh-Taylor instability of the air and Helium which are confined within a closed box of width $10cm$ and height $40cm$. The density and the viscosity are $1.225 \times 10^{-3} g/cm^3$ and $1.776 \times 10^{-4} g/(cm \cdot s)$ respectively for the air, and $1.655 \times 10^{-4} g/cm^3$ and $1.941 \times 10^{-4} g/(cm \cdot s)$ for the Helium. In this case the ratios of the density and the dynamical viscosity are $\rho_{air}/\rho_{helium} = 7.4$ and $\mu_{air}/\mu_{helium} = 0.13$, respectively. The gravitational acceleration is $980 cm/s^2$.

We consider the flat initial interface and the perturvation is supplied by specifying the vertical component of the initial velocity u_0 at the interface to be $A\cos(\frac{2\pi x_1}{10})$ for $0 \leq x_1 \leq 10$, where A is the amplitude of the purturbation. Evidently this initial condition is adapted to the incompressible condition. For the initial function of the pseudo-density function, we set as follows :

$$\phi_0(x) = \begin{cases} -1 & (0 \leq x_2 < 20 - \epsilon), \\ \sin\frac{\pi}{2\epsilon}(x_2 - 20) & (20 - \epsilon \leq x_2 \leq 20 + \epsilon), \\ 1 & (20 + \epsilon < x_2 \leq 40). \end{cases} \tag{7.1}$$

All computaions were performed on a adaptive-like finite element mesh with 2305 nodes and 1108 elements. The time step is set to $\tau = 1.0 \times 10^{-4}$. Figure 1 shows the finite element mesh used in our computations and the profile of the initial function $\phi_0(x)$ of the pseudo-density function.

We present a numerical example in the case of $\sigma = 0$. Figure 2 shows the profile of the pseudo-density function and interface at different times. From these figures

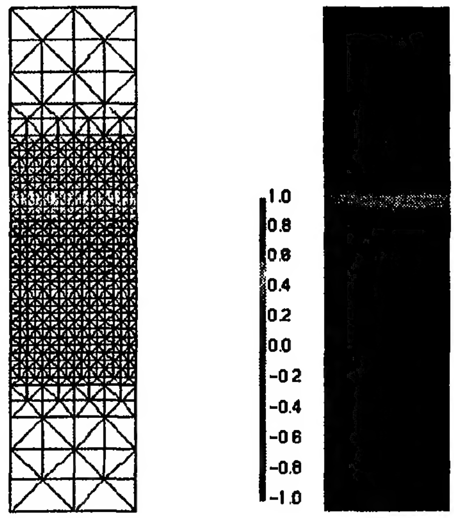

Figure 1: Finite element mesh and profile of $\varphi_0(x)$

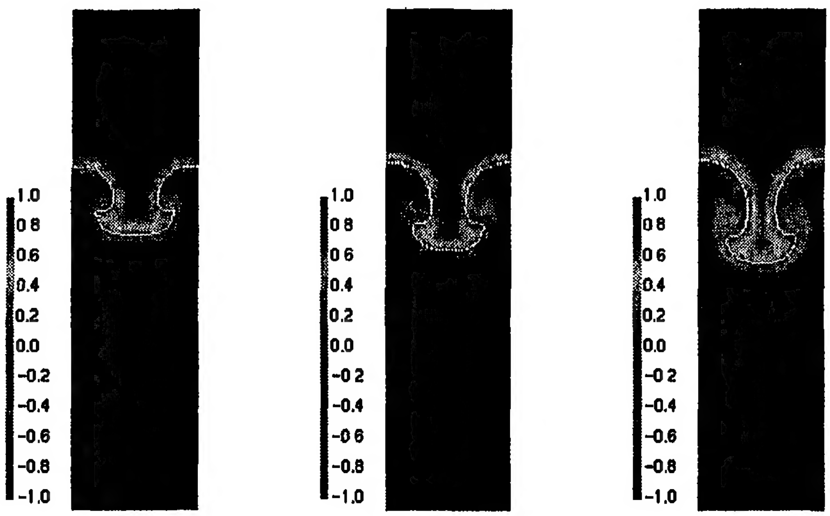

Figure 2: Profiles of the pseudo-density function and interface at $t = 0.02,\ 0.03,\ 0.04$.

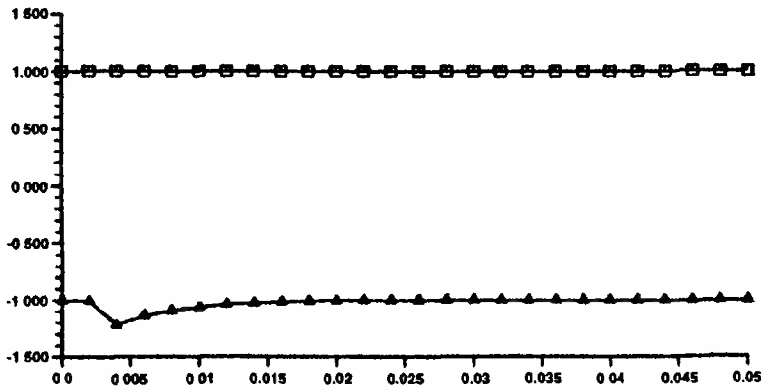

Figure 3: Time history of maximum and minimum values of the pseudo-density function

it is easily seen that the interface between two fluids formes like a *mushroom cap*, which is the typical feature of the Rayleigh-Taylor instability. Figure 3 shows the time history of the maximum and minimum values of the pseudo-density function. The time history of the total mass of *fluid#2*(Helium) is plotted in Figure 4. Here the initial total mass of Helium is 0.609. From these figures we can observe that the conservation of total mass is almost achieved within a small tolerance. Since our scheme for the pure advection equation has no re-initialization procedure, it needs some re-initialization techniques to obtain an excellent conservation of total mass. Furthermore, Figure 5 shows the velocity vector field at different times.

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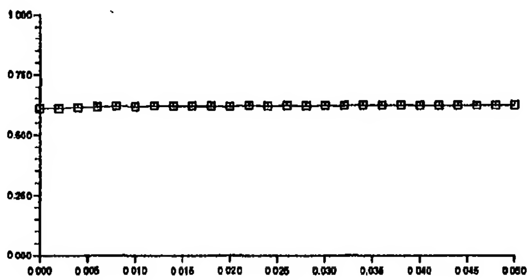

Figure 4: Time history of the total mass of Helium

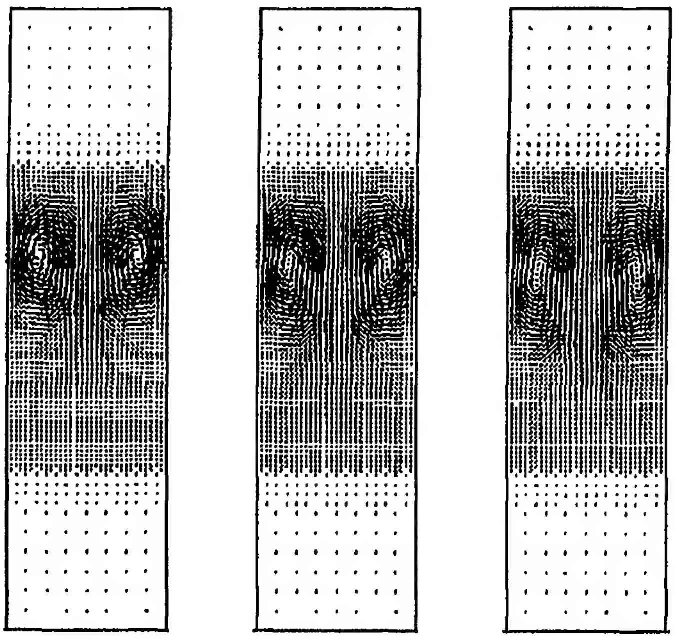

Figure 5: Velocity vector fields at $t = 0.02,\ 0.03,\ 0.04$.

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